

Remarks on the thermodynamics and the vacuum energy of a quantum Maxwell gas on compact and closed manifolds

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Abstract

The quantum Maxwell theory at finite temperature at equilibrium is studied on compact and closed manifolds in both the functional integral- and Hamiltonian formalism. The aim is to shed some light onto the interrelation between the topology of the spatial background and the thermodynamic properties of the system. The quantization is not unique and gives rise to inequivalent quantum theories which are classified by θ -vacua. Based on explicit parametrizations of the gauge orbit space in the functional integral approach and of the physical phase space in the canonical quantization scheme, the Gribov problem is resolved and the equivalence of both quantization schemes is elucidated. Using zeta-function regularization the free energy is determined and the effect of the topology of the spatial manifold on the vacuum energy and on the thermal gauge field excitations is clarified. The general results are then applied to a quantum Maxwell gas on a n -dimensional torus providing explicit formulae for the main thermodynamic functions in the low- and high temperature regimes, respectively.

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1 Introduction

Quantum field theories at finite temperature have been intensively studied during the last years [1, 2]. Prominent examples which stimulate this interest are the study of matter formation in the early stage of the universe, the description of the quark gluon plasma in the context of the AdS/CFT correspondence [3] and the current intense and partly controversial discussion concerning the Casimir effect [4] at finite temperature (for a review see [5, 6, 7, 8, 9, 10]). From a general perspective, the *thermal Casimir effect* can be regarded as a deviation of the vacuum energy and the energy of thermal excitations of a quantum field caused by the presence of external constraints. These constraints may be imposed either by real material boundaries or by topologically non-trivial manifolds on which the quantum fields reside. In any case, the

modes of the quantum fields are correspondingly restricted, affecting both the vacuum energy and the thermodynamic functions of the system.

In particular, it is the relation between the topology of the spatial background and the thermodynamic properties of gauge fields which in our opinion deserves closer attention and is the main motivation for the present paper. Our purpose is to study quantum Maxwell theory at finite temperature at equilibrium on a n -dimensional compact, closed and connected manifold X , which represents the spatial background. We perform the analysis in both the functional integral and Hamiltonian approach and derive the expression for the regularized free energy.

Let us now motivate our intention of the present paper in more detail: Quantum fields at finite temperature on general manifolds with and without a boundary have been considered for many years [11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25]. (Here we have selected a list of references which are the most relevant ones in the context we are interested in). Most of the research is in fact dedicated to the study of scalar fields. It is often argued, however, that the thermodynamic functions for a photon gas could be obtained directly from those for a massless scalar gas just by multiplication with the number of independent polarization states. We will show explicitly that this statement is not true in general. This requires a critical review of the concept which underlies the description of finite temperature gauge fields.

In the functional integral formulation the thermal partition function of a bosonic systems at finite temperature T is obtained by integrating the classical action over all fields periodic in the (imaginary) time coordinate with period $\beta := \frac{1}{k_B T}$, where k_B is the Boltzmann constant [1, 2]. Geometrically this corresponds to a quantum field theory of bosons on the product space $\mathbb{T}_\beta^1 \times X$, where \mathbb{T}_β^1 denotes the circle (1-torus) with circumference β .

Evidently, the partition function plays a major role in the finite temperature context, since it serves as basic quantity from which the thermodynamic functions are derived. Hence particular care has to be taken in order to obtain the correct measure for the functional integral representation of the partition function. This issue has been explicitly discussed in the seminal papers [26, 27] for quantum Maxwell theory in the covariant gauge in the non-compact but topologically trivial case $X = \mathbb{R}^n$. Following the Faddeev-Popov approach, the corresponding Faddeev-Popov determinant which appears as a field independent but temperature dependent multiplicative factor after gauge fixing must be retained in the partition function. Compared to the zero-temperature case, this factor is necessary to compensate the contributions caused by the unphysical degrees of freedom.

But what happens if the thermal gauge theory suffers from Gribov ambiguities, which prevent the existence of a global and unique solution of the gauge-fixing condition, and thus the existence of any globally defined measure on the space of thermal gauge fields?

It is a general result that abelian gauge theories suffer from the Gribov problem whenever the gauge fields reside on a non-simply connected, compact manifold [28, 29]. However, this is precisely the situation one encounters in the finite temperature context since the base manifold is the product space $\mathbb{T}_\beta^1 \times X$. Thus even for topologically trivial spatial manifolds X , this problem does exist. This fact had been recognized long time ago [30, 31] in the case of quantum Maxwell theory at finite temperature on a 3-sphere ($X = \mathbb{S}^3$) by solving the corresponding gauge-fixing condition. The main result was that the Faddeev-Popov operator (i.e. the zeroth order Laplace operator on $\mathbb{T}_\beta^1 \times \mathbb{S}^3$) possesses temperature dependent zero-modes. Based on the results of Ref. [26] it was then argued, but not proved, that the Faddeev-Popov formula for the partition function is valid as long as the zero modes are ruled out from the domain of the Faddeev-Popov operator. Although this argument is valid for $X = \mathbb{S}^3$, we will show, however, that restriction to

the non-zero modes alone is not sufficient to give the correct partition function in the general case.

In this paper the construction of the partition function and thus the Gribov problem will be tackled in a different way. As a by-product the relation between the occurrence of the zero modes and the Gribov ambiguities is elucidated, too. Thereby we will focus on the following four objectives:

- Resolve the Gribov problem and construct a reasonable functional integral representation of the free energy including a finite expression for the vacuum energy using zeta function regularization.
- Analyze the relation between the functional integral quantization and the Hamiltonian (canonical) quantization at finite temperature.
- Study the impact of the topology and geometry of the spatial manifold X on the thermodynamic properties of the system.
- Determine the thermodynamic functions for a photon gas confined to a n -dimensional torus (i.e. $X = \mathbb{T}^n$).

To our knowledge no comprehensive treatment of the various aspects of the Gribov problem in the finite temperature context has been given in detail so far.

This paper is organized as follows: In Section 2 the geometrical structure of the space of thermal gauge fields is presented. It is shown that there exist gauge transformations not connected to unity. As a consequence the bundle of thermal gauge fields over the space of gauge inequivalent fields is not trivializable, which in physical terms is expressed by the statement that the theory suffers from Gribov ambiguities.

Section 3 is devoted to the construction of a functional integral representation of the partition function in the space of thermal gauge fields. In order to circumvent the Gribov problem we will apply a method which has been introduced some time ago in the stochastic quantization scheme of Yang-Mills theory [32, 33]. For abelian gauge theories this procedure was elaborated recently in Ref. [29], where it was also explicitly shown why the conventional Faddeev-Popov procedure fails on non-simply connected manifolds.

The idea is to select a family of functional integral measures on the space of thermal gauge fields, whose domains of definition are determined by local gauge fixing submanifolds and which are integrable along the orbits of the gauge group. Finally these local measures are glued together in such a way that the physical relevant objects become independent of the chosen regularization of the gauge group and of the particular way this gluing was provided. The redundant gauge degrees of freedom are taken into account by factoring out the regularized volume of the full gauge group. This will be a slight extension of the original procedure [32], where only those gauge transformations were ruled out, which act freely on the space of thermal gauge fields. As a consequence of this reduction process, an additional topological factor apart from the conventional Faddeev-Popov determinant will appear in the functional integral measure. Whereas this factor may be neglected in the zero-temperature case, we will see that it contributes to the thermodynamical properties of the system and must be retained in the finite temperature context. Additionally, different topological sectors may exist on a general manifold X . In order to include all relevant thermal contributions, these sectors have to be considered as well when constructing the thermal partition function.

In accordance to the non-trivial topology of the space of inequivalent thermal gauge fields, the quantization is not unique. In the functional integral approach this fact is usually taken into account by adding a total derivative to the classical Maxwell Lagrangian. This term does not alter the classical equations of motion but contributes in the quantized version whenever the integration runs over topologically non-trivial field configurations. In our case this additional action will be shown to be parametrized by a constant vector $\vec{\theta} \in \mathbb{R}^{b_1(X)}$, where $b_1(X)$ is the first Betti number of X . The functional integral can be solved exactly and by using zeta function regularization technique [34, 35, 36] we obtain a closed expression for the free energy of the system. This is achieved by expressing the zeta-function of the Laplace operators on $\mathbb{T}_\beta^1 \times X$ in terms of the zeta function associated with the corresponding Laplace operators on X .

The quantum Maxwell theory at finite temperature is discussed from the canonical (Hamiltonian) point of view in Section 4. We will determine the true phase space \mathcal{P} and provide an explicit parametrization of this manifold. Since \mathcal{P} turns out to be non-simply connected, there exist inequivalent quantum theories which are classified by unitary irreducible representations of $\pi_1(\mathcal{P})$. These are labelled by the non-integer values $\vec{\theta} \in \mathbb{R}^{b_1(X)}$. We will determine the Hamilton operator and calculate its spectrum. The finite total vacuum energy of the system turns out to be the sum of the vacuum energy of the transverse gauge fields, which is regularized by adopting the minimal subtraction scheme [35], and the energy of the θ -states. The latter vanishes whenever $\vec{\theta} \in \mathbb{Z}^{b_1(X)}$. The free energy is derived within the Hamiltonian scheme and the agreement with the functional integral formalism is explicitly proved.

Finally, the high-temperature asymptotic expansion is calculated in terms of the heat kernel (Seeley) coefficients giving rise to finite size and topological contributions to the familiar Stefan-Boltzmann black-body radiation law.

The scaling property of the system under constant scale transformations is analyzed in Section 5. Due to the regularization ambiguities of the infinite vacuum energy and the non-trivial cohomology of X , the free energy does no longer transform homogeneously. This leads to a modified equation of state.

In Section 6 we discuss the example of finite temperature quantum Maxwell theory on the n -torus $X = \mathbb{T}^n$. The low- and high temperature expansions are calculated for the main thermodynamic functions in terms of the Epstein zeta function [37, 38] and the Riemann Theta function. Due to the θ -term the ground state is degenerate. In the zero-temperature limit the entropy converges to the logarithm of the degree of degeneracy, which proves explicitly the validity of the 3rd law of thermodynamics (i.e. Nernst theorem).

Necessary results on Riemann Theta functions and Epstein zeta functions are summarized in the Appendix. The conventions $c = \hbar = k_B = 1$ are used.

2 The configuration space of thermal gauge fields

In this section we will consider the geometrical structure of the space of thermal gauge fields. These results are based on Ref. [29], where a detailed analysis can be found.

Geometrically, the finite temperature quantum Maxwell theory at equilibrium at finite temperature $1/\beta$ is regarded as gauge theory on the product manifold $\mathbb{T}_\beta^1 \times X$ with product metric $g = \gamma_\beta \oplus \gamma$. Here \mathbb{T}_β^1 denotes the 1-torus \mathbb{T}^1 equipped with the temperature dependent Riemannian metric $\gamma_\beta = \beta^2 dt \otimes dt$ (t is the local coordinate on \mathbb{T}^1) and the spatial background X is a n -dimensional compact, connected, oriented and closed Riemannian manifold equipped with a

fixed metric γ . The corresponding volume forms on \mathbb{T}_β^1 and X are denoted by $vol_{\mathbb{T}_\beta^1}$ and vol_X , respectively.

Let $(Q, \pi_Q, \mathbb{T}_\beta^1 \times X)$ be a principal $U(1)$ -bundle over $\mathbb{T}_\beta^1 \times X$ with projection π_Q . The space of thermal gauge fields is identified with the C^∞ -Hilbert manifold \mathcal{A}^Q of all connections of Q . The space of \mathbb{R} -valued k -forms on $\mathbb{T}_\beta^1 \times X$, denoted by $\Omega^k(\mathbb{T}_\beta^1 \times X)$, is equipped with the L^2 -inner product

$$\langle v_1, v_2 \rangle_g = \int_{\mathbb{T}_\beta^1 \times X} v_1 \wedge \star_g v_2 \quad v_1, v_2 \in \Omega^k(\mathbb{T}_\beta^1 \times X), \quad (2.1)$$

where \star_g is the Hodge star operator on $\mathbb{T}_\beta^1 \times X$, satisfying $\star_g^2 = (-1)^{k(n+1-k)}$ when acting on k -forms. The co-differential $d_k^* = (-1)^{(n+1)(k+1)+1} \star_g d_{n+1-k} \star_g : \Omega^k(\mathbb{T}_\beta^1 \times X) \rightarrow \Omega^{k-1}(\mathbb{T}_\beta^1 \times X)$ gives rise to the Laplacian operator $\Delta_k^{\mathbb{T}_\beta^1 \times X} = d_{k+1}^* d_k + d_{k-1} d_k^*$. Let $\mathcal{H}^k(\mathbb{T}_\beta^1 \times X)$ denote the space of \mathbb{R} -valued harmonic k -forms on $\mathbb{T}_\beta^1 \times X$ and $\mathcal{H}^k(\mathbb{T}_\beta^1 \times X)^\perp$ its orthogonal complement. Then we can define the Green's operator by

$$G_k^{\mathbb{T}_\beta^1 \times X} : \Omega^k(\mathbb{T}_\beta^1 \times X) \rightarrow \mathcal{H}^k(\mathbb{T}_\beta^1 \times X)^\perp, \quad G_k^{\mathbb{T}_\beta^1 \times X} := (\Delta_k^{\mathbb{T}_\beta^1 \times X} |_{\mathcal{H}^k(\mathbb{T}_\beta^1 \times X)^\perp})^{-1} \circ \Pi^{\mathcal{H}^k(\mathbb{T}_\beta^1 \times X)^\perp}, \quad (2.2)$$

where $\Pi^{\mathcal{H}^k(\mathbb{T}_\beta^1 \times X)^\perp}$ is the projection onto $\mathcal{H}^k(\mathbb{T}_\beta^1 \times X)^\perp$. By construction one immediately obtains $\Delta_k^{\mathbb{T}_\beta^1 \times X} \circ G_k^{\mathbb{T}_\beta^1 \times X} = G_k^{\mathbb{T}_\beta^1 \times X} \circ \Delta_k^{\mathbb{T}_\beta^1 \times X} = \Pi^{\mathcal{H}^k(\mathbb{T}_\beta^1 \times X)^\perp}$. The dimension of $\mathcal{H}^k(\mathbb{T}_\beta^1 \times X)$ is given by the k -th Betti number $b_k(\mathbb{T}_\beta^1 \times X) = b_k(X) + b_{k-1}(X)$. Here $b_k(X)$ is the k -th Betti number of X .

Let $\mathfrak{t}^1 := \sqrt{-1} \mathbb{R}$ denote the Lie-algebra of $U(1)$. The space of thermal gauge fields \mathcal{A}^Q is an affine space modelled on $\Omega^1(\mathbb{T}_\beta^1 \times X) \otimes \mathfrak{t}^1$ and can be equipped with the following metric

$$\hat{g}(w_1, w_2) = \langle \frac{w_1}{\sqrt{-1}}, \frac{w_2}{\sqrt{-1}} \rangle_g \quad (2.3)$$

where $w_1, w_2 \in \Omega^1(\mathbb{T}_\beta^1 \times X) \otimes \mathfrak{t}^1$. Let $vol_{\mathcal{A}^Q}^{\hat{g}}$ denote the induced (formal) volume form on \mathcal{A}^Q .

The gauge group \mathcal{G} is defined as the group of vertical bundle automorphisms on Q and can be identified with the Hilbert Lie-Group $C^\infty(\mathbb{T}_\beta^1 \times X; U(1))$ of differentiable maps. The corresponding Lie algebra is $\mathfrak{G} = \Omega^0(\mathbb{T}_\beta^1 \times X; \mathfrak{t}^1)$. Since there is at least one non-contractible loop in $\mathbb{T}_\beta^1 \times X$, the gauge group contains gauge transformations which cannot be connected to the unity. These transformations are classified by a non-trivial $\pi_0(\mathcal{G})$ and cannot be generated by taking the exponential of imaginary-valued functions on $\mathbb{T}_\beta^1 \times X$. Under an arbitrary gauge transformation $v \in \mathcal{G}$, the thermal gauge fields transform according to

$$A \mapsto A^v = A + (\pi_Q^* v)^* \vartheta^{U(1)} \quad v \in \mathcal{G}, \quad (2.4)$$

where $\vartheta^{U(1)} \in \Omega^1(\mathbb{T}^1; \mathfrak{t}^1)$ is the Maurer Cartan form on $U(1)$. (For notational convenience we shall not distinguish between $\pi_Q^* v$ and v .) The isotropy group of the non-free action (2.4) is $U(1)$, which corresponds to the subgroup of constant gauge transformations. However, the quotient group $\mathcal{G}_* := \mathcal{G}/U(1)$ provides a free action onto \mathcal{A}^Q giving rise to a smooth manifold $\mathcal{M}_*^Q = \mathcal{A}^Q/\mathcal{G}_*$ which represents the reduced configuration space in the functional integral picture.

In order to fix notation, let $\mathcal{H}_{\mathbb{Z}}^k(X)$ denote the abelian group of harmonic \mathbb{R} -valued differential k -forms with integer periods along $Z_k(X; \mathbb{Z})$, where $Z_k(X; \mathbb{Z})$ is the subcomplex of all closed smooth singular k -cycles on X with integer coefficients.

In our case we can choose a set of 1-cycles $c_i \in Z_1(X, \mathbb{Z})$, $i = 1, \dots, b_1(X)$, whose corresponding homology classes $[c_i]$ provide a Betti basis of $H_1(X; \mathbb{Z})$. Let $\rho_j^{(n-1)} \in \mathcal{H}_{\mathbb{Z}}^{n-1}(X)$ be a basis associated to the Betti basis $[c_i]$ via the Poincare duality, where $j = 1, \dots, b_{n-1}(X)$. A dual basis $\rho_i^{(1)} \in \mathcal{H}_{\mathbb{Z}}^1(X)$ can be adjusted such that $\int_{c_j} \rho_i^{(1)} = \int_X \rho_i^{(1)} \wedge \rho_j^{(n-1)} = \delta_{ij}$. Let $(h_X^{(1)})_{ij} = \langle \rho_i^{(1)}, \rho_j^{(1)} \rangle_{\gamma} = \int_X \rho_i^{(1)} \wedge \star_{\gamma} \rho_j^{(1)}$ denote the induced metric on $\mathcal{H}^1(X)$. Notice that $\star_{\gamma} \rho_i^{(1)} = \sum_{j=1}^{b_1(X)} (h_X^{(1)})_{ij} \rho_j^{(n-1)}$. The harmonic 1-forms $(pr_{\mathbb{T}_{\beta}^1}^* \varrho^{(1)}, pr_X^* \rho_1^{(1)}, \dots, pr_X^* \rho_{b_1(X)}^{(1)})$ generate $\mathcal{H}_{\mathbb{Z}}^1(\mathbb{T}_{\beta}^1 \times X)$, where $\varrho^{(1)} = \frac{1}{\beta} \text{vol}_{\mathbb{T}_{\beta}^1}$. Here $pr_{\mathbb{T}_{\beta}^1}: \mathbb{T}_{\beta}^1 \times X \rightarrow \mathbb{T}_{\beta}^1$ and $pr_X: \mathbb{T}_{\beta}^1 \times X \rightarrow X$ are the natural projections.

Let $(s_0, x_0) \in \mathbb{T}_{\beta}^1 \times X$ be an arbitrary but fixed point. The gauge group \mathcal{G}_* is equivalently characterized as particular subgroup $\{v \in \mathcal{G} | v(s_0, x_0) = 1\}$ of \mathcal{G} . The corresponding Lie algebra $\mathfrak{G}_* = \mathfrak{G}/\mathfrak{t}^1$ contains all those $\xi \in \mathfrak{G}$ such that $\xi(s_0, x_0) = 0$. We use this alternative description to rewrite any (restricted) gauge transformation as product of an infinitesimal- and large gauge transformation. For this we construct an isomorphism $\kappa: \mathfrak{G}_* \times \mathcal{H}_{\mathbb{Z}}^1(\mathbb{T}_{\beta}^1 \times X) \rightarrow \mathcal{G}_*$ by

$$\kappa(\xi, \alpha)(s, x) = \exp \xi(s, x) \cdot \exp \left(2\pi\sqrt{-1} \int_{c_{(s,x)}} \alpha \right). \quad (2.5)$$

Here $c_{(s,x)}: [0, 1] \rightarrow \mathbb{T}_{\beta}^1 \times X$ is an arbitrary path in $\mathbb{T}_{\beta}^1 \times X$ connecting (s_0, x_0) with (s, x) . As a result, (2.4) can be rewritten into the form

$$A \mapsto A + d_0 \xi + 2\pi\sqrt{-1} \left(m_0 pr_{\mathbb{T}_{\beta}^1}^* \varrho^{(1)} + \sum_{j=1}^{b_1(X)} m_j pr_X^* \rho_j^{(1)} \right), \quad \xi \in \mathfrak{G}_*, \quad m_0, m_j \in \mathbb{Z}. \quad (2.6)$$

For any choice of an arbitrary but fixed background gauge field $A_0 \in \mathcal{A}^Q$ there exists a smooth surjective map $\pi_{\mathcal{M}_*^Q}^{A_0}: \mathcal{M}_*^Q \rightarrow \mathbb{T}^{1+b_1(X)}$, defined by

$$\pi_{\mathcal{M}_*^Q}^{A_0}([A]) = \left(e^{\int_{\mathbb{T}_{\beta}^1 \times \{x_0\}} (A - A_0)}, e^{\int_{\{s_0\} \times c_1} (A - A_0)} \dots, e^{\int_{\{s_0\} \times c_{b_1(X)}} (A - A_0)} \right). \quad (2.7)$$

According to the general result proved in [29], we thus obtain the following two propositions, which summarize the topological structure of the space of thermal gauge fields:

Proposition 1. $(\mathcal{A}^Q, \pi_{\mathcal{A}^Q}, \mathcal{M}_*^Q)$ is a non trivializable flat principal \mathcal{G}_* -bundle over \mathcal{M}_*^Q with projection $\pi_{\mathcal{A}^Q}$. \square

Proposition 2. For an arbitrary but fixed connection $A_0 \in \mathcal{A}^Q$, $\pi_{\mathcal{M}_*^Q}^{A_0}: \mathcal{M}_*^Q \rightarrow \mathbb{T}^{1+b_1(X)}$ admits the structure of a trivializable vector bundle over $\mathbb{T}^{1+b_1(X)}$ with projection $\pi_{\mathcal{M}_*^Q}^{A_0}$ and typical fiber $\mathcal{N} := \text{imd}_2^* \otimes \mathfrak{t}^1$. If $A'_0 \in \mathcal{A}^Q$ is a different background connection, then $\pi_{\mathcal{M}_*^Q}^{A'_0}: \mathcal{M}_*^Q \rightarrow \mathbb{T}^{1+b_1(X)}$ is an isomorphic vector bundle. \square

As a result, the manifolds \mathcal{A}^Q and $\mathbb{T}^{1+b_1(X)} \times \mathcal{N} \times \mathcal{G}_*$ are locally diffeomorphic. An explicit expression for these local diffeomorphisms can be given as follows: Let us introduce the two

contractible open sets $V_{a_j=1} = \mathbb{T}^1 \setminus \{\text{northernpole}\}$ and $V_{a_j=2} = \mathbb{T}^1 \setminus \{\text{southernpole}\}$, which cover the j -th 1-torus $\mathbb{T}^1 \subset \mathbb{T}^{1+b_1(X)}$. Then

$$\mathcal{V} = \{V_a := V_{a_0} \times \cdots \times V_{a_j} \times \cdots \times V_{a_{b_1(X)}} \mid a := (a_0, \dots, a_j, \dots, a_{b_1(X)}), \quad a_j \in \{1, 2\}\}, \quad (2.8)$$

is an open cover of $\mathbb{T}^{1+b_1(X)}$. Hence the family of open sets $U_a^{A_0} := (\pi_{\mathcal{M}_*^Q}^{A_0})^{-1}(V_a)$ provides a finite open cover $\mathcal{U}^{A_0} = \{U_a^{A_0}\}$ of \mathcal{M}_*^Q . Any two local sections $\sigma_{a_j} : V_{a_j} \rightarrow \mathbb{R}^1$ of the universal covering bundle $\mathbb{R}^1 \rightarrow \mathbb{T}^1$ generate $2^{1+b_1(X)}$ local sections $\sigma_a : V_a \subset \mathbb{T}^{1+b_1(X)} \rightarrow \mathbb{R}^{1+b_1(X)}$, given by $\sigma_a = (\sigma_{a_0}, \dots, \sigma_{a_{b_1(X)}})$. It can be shown that the maps $\chi_a^{A_0} : V_a \times \mathcal{N} \times \mathcal{G}_* \rightarrow \pi_{\mathcal{A}^Q}^{-1}(U_a^{A_0}) \subseteq \mathcal{A}^Q$, defined by

$$\chi_a^{A_0}(z_0, \dots, z_{b_1(X)}, \tau, v) = A_0 + 2\pi\sqrt{-1} \left(\sigma_{a_0}(z_0) pr_{\mathbb{T}_\beta^1}^* \varrho + \sum_{i=1}^{b_1(X)} \sigma_{a_i}(z_i) pr_X^* \rho_i^{(1)} \right) + \tau + v^* \vartheta^{U(1)}. \quad (2.9)$$

provide an appropriate family of local diffeomorphisms.

3 The free energy in the functional integral approach

3.1 The construction principle

In this section we want to construct a functional integral representation of the (thermal) partition function for the photon gas. The Maxwell fields are governed by the classical action

$$S_{inv}(A) = \frac{1}{2} \|F_A\|^2 = \frac{1}{2} \hat{g}(F_A, F_A). \quad (3.1)$$

In the previous section it has been shown that the reduced configuration space is the non-simply connected manifold \mathcal{M}_*^Q , since $\pi_1(\mathcal{M}_*^Q) = \pi_0(\mathcal{G}_*) \cong \mathbb{Z}^{1+b_1(X)}$. It is well known that ambiguities arise in the definition of the vacuum if the corresponding configuration space is non-simply connected. In the functional integral approach this fact is usually taken into account by adding a so-called topological- or theta action, denoted by S_θ , to the gauge invariant classical action.

To each arbitrary but fixed vector $\vec{\theta} = (\theta_1, \dots, \theta_{b_1(X)}) \in \mathbb{R}^{b_1(X)}$ we associate the harmonic 1-form $\underline{\theta} := 2\pi\sqrt{-1} \sum_{i,j=1}^{b_1(X)} (h_X^{(1)})_{ij}^{-1} \theta_i \rho_j^{(1)} \in \mathcal{H}^1(X) \otimes \mathfrak{t}^1$. We propose the following action

$$\begin{aligned} S_\theta(A) &:= \frac{1}{2\pi\sqrt{-1}} \hat{g}(F_A, \star_g pr_X^* \star_\gamma \underline{\theta}) \\ &= - \sum_{i=1}^{b_1(X)} \theta_i \int_{\mathbb{T}_\beta^1 \times X} F_A \wedge pr_X^* \rho_i^{(n-1)}, \end{aligned} \quad (3.2)$$

which is indeed topological since it is independent of the metric of $\mathbb{T}_\beta^1 \times X$. This specific choice will become more transparent in the next section when the Hamiltonian approach is considered. For the total action we take now $S_{tot} := S_{inv} + S_\theta$ instead of S_{inv} alone. Since $\frac{\delta S_\theta}{\delta A} = 0$, the inclusion of this additional term does not change the classical equations of motion. The theta

action S_θ is imaginary because our context is the finite-temperature Euclidean regime where time is compactified. In fact, analytic continuation of (3.2) onto the real time axis is necessary to obtain the correct real-valued theta-term in the Hamiltonian formulation on space-time $\mathbb{R} \times X$.

The total configuration space, denoted by $\mathcal{A}_{\mathbb{T}_\beta^1 \times X}$, is the disjoint union

$$\mathcal{A}_{\mathbb{T}_\beta^1 \times X} = \bigsqcup_Q \mathcal{A}^Q \quad (3.3)$$

over equivalence classes of principal $U(1)$ -bundles Q over the base manifold $\mathbb{T}_\beta^1 \times X$. Each class is labelled by its first Chern-class $c_1(Q) \in H^2(\mathbb{T}_\beta^1 \times X; \mathbb{Z}) \cong H^1(X; \mathbb{Z}) \oplus H^2(X; \mathbb{Z})$.

In the conventional functional integral formulation the integration of the functional $e^{-S_{tot}} vol_{\mathcal{A}^Q}^{\hat{g}}$ over \mathcal{A}^Q would become infinite due to the gauge invariance of the classical action. Our aim is to give this integral a well-defined meaning by damping the contributions from the gauge dependent degrees of freedom. The non-trivial bundle structure $\mathcal{A}^Q \xrightarrow{\pi_{\mathcal{A}^Q}} \mathcal{M}_*^Q$ allows only for a local damping procedure, which, however, can be patched together using a partition of unity in the end.

The map $[\xi] \mapsto d_1^* d_0 G_0^{\mathbb{T}_\beta^1 \times X}(\xi)$ provides an isomorphism $\mathfrak{G}_* \cong im d_1^* \otimes \mathfrak{t}^1$. Notice that \mathcal{G} is a trivializable principal $U(1)$ -bundle over \mathcal{G}_* , whose global trivialization is given by the diffeomorphism $\hat{\sigma}(u) = (uu(s_0, x_0)^{-1}, u(s_0, x_0))$ where $u \in \mathcal{G}$. Let $\vartheta^{\mathcal{G}} \in \Omega^1(\mathcal{G}) \otimes \mathfrak{G}$ and $\vartheta^{\mathcal{G}_*} \in \Omega^1(\mathcal{G}_*) \otimes \mathfrak{G}_*$ denote the Maurer Cartan forms on \mathcal{G} and \mathcal{G}_* , respectively. Formally one can introduce the natural metrics $\Upsilon^{\mathcal{G}} = \hat{g}(\vartheta^{\mathcal{G}}(\cdot), \vartheta^{\mathcal{G}}(\cdot))$ and $\Upsilon^{\mathcal{G}_*} = \hat{g}(\vartheta^{\mathcal{G}_*}(\cdot), \vartheta^{\mathcal{G}_*}(\cdot))$ on these two gauge groups. Let $vol_{\mathcal{G}}$ and $vol_{\mathcal{G}_*}$ denote the induced left-invariant volume forms. Furthermore $U(1)$ is equipped with the standard flat metric giving rise to the volume form $vol_{U(1)} = (\sqrt{-1})^{-1} \vartheta^{U(1)}$. With respect to our parametrization of $U(1)$, the corresponding volume is $\int_{U(1)} vol_{U(1)} = 2\pi$.

Let us now introduce three real-valued regularizing functions $\Lambda_{reg}^{\mathcal{G}} \in C^\infty(\mathcal{G})$, $\Lambda_{reg}^{\mathcal{G}_*} \in C^\infty(\mathcal{G}_*)$ and $\Lambda_{reg}^{U(1)} \in C^\infty(U(1))$ in such a way that the following correspondingly regularized group volumes

$$\begin{aligned} Vol(\mathcal{G}; e^{-\Lambda_{reg}^{\mathcal{G}}}) &:= \int_{\mathcal{G}} vol_{\mathcal{G}} e^{-\Lambda_{reg}^{\mathcal{G}}} < \infty \\ Vol(\mathcal{G}_*; e^{-\Lambda_{reg}^{\mathcal{G}_*}}) &:= \int_{\mathcal{G}_*} vol_{\mathcal{G}_*} e^{-\Lambda_{reg}^{\mathcal{G}_*}} < \infty \\ Vol(U(1); e^{-\Lambda_{reg}^{U(1)}}) &:= \int_{U(1)} vol_{U(1)} e^{-\Lambda_{reg}^{U(1)}} \end{aligned} \quad (3.4)$$

become finite. Evidently, a regularization of $U(1)$ is not necessary (of course $\Lambda_{reg}^{U(1)} = 0$ would be sufficient) but we will use this freedom later on to absorb irrelevant multiplicative factors in the partition function.

In order to relate these volumes, we consider the differential of $\hat{\sigma}^{-1}$, which gives

$$T\hat{\sigma}^{-1}(\tilde{\xi}, w) = Tr_z^{\mathcal{G}} \left(\tilde{\xi} + \frac{d}{dt} \Big|_{t=0} v \cdot e^{t\vartheta^{U(1)}(w)} \right), \quad \tilde{\xi} \in T_v \mathcal{G}_*, \quad w \in T_z U(1), \quad (3.5)$$

where $r_z^{\mathcal{G}}$ denotes the right multiplication on \mathcal{G} by $z \in U(1)$. Since $\vartheta^{\mathcal{G}}(\tilde{\xi}) = \vartheta^{\mathcal{G}_*}(\tilde{\xi}) \in im d_1^* \otimes \mathfrak{t}^1$ the induced metrics are related by

$$(\hat{\sigma}^{-1})^* \Upsilon^{\mathcal{G}} = \Upsilon^{\mathcal{G}_*} - \beta V \vartheta^{U(1)}(\cdot) \vartheta^{U(1)}(\cdot), \quad (3.6)$$

where $V := \int_X \text{vol}_X$ is the volume of X with respect to the metric γ . For the volume forms, this finally implies

$$(\hat{\sigma}^{-1})^* \text{vol}_{\mathcal{G}} = (\beta V)^{\frac{1}{2}} \text{vol}_{\mathcal{G}_*} \wedge \text{vol}_{U(1)}. \quad (3.7)$$

Let us take the natural choice $\Lambda_{reg}^{\mathcal{G}} := \pi_{\mathcal{G}}^* \Lambda_{reg}^{\mathcal{G}_*} + \hat{\sigma}_{(s_0, x_0)}^* \Lambda_{reg}^{U(1)}$, where $\pi_{\mathcal{G}} : \mathcal{G} \rightarrow \mathcal{G}_*$ is the projection and $\hat{\sigma}_{(s_0, x_0)}(u) := u(s_0, x_0)$. Using (3.4) it follows that

$$\text{Vol}(\mathcal{G}; e^{-\Lambda_{reg}^{\mathcal{G}}}) = (\beta V)^{\frac{1}{2}} \text{Vol}(\mathcal{G}_*; e^{-\Lambda_{reg}^{\mathcal{G}_*}}) \text{Vol}(U(1); e^{-\Lambda_{reg}^{U(1)}}). \quad (3.8)$$

An appropriate choice for the regularizing function is provided by

$$\Lambda_{reg}^{\mathcal{G}_*}(v) = \frac{1}{2} \|d_1^*(v^* \vartheta^{U(1)})\|^2 + \frac{1}{2} \|\Pi^{\mathcal{H}^1(\mathbb{T}_{\beta}^1 \times X)}(v^* \vartheta^{U(1)})\|^2, \quad (3.9)$$

where $\Pi^{\mathcal{H}^1(\mathbb{T}_{\beta}^1 \times X)}$ is the projector onto $\mathcal{H}^1(\mathbb{T}_{\beta}^1 \times X)$ [29]. According to (2.5) and with respect to the chosen Betti-basis, each $v \in \mathcal{G}_*$ is uniquely represented by a pair $(\xi, \vec{m}) \in (\text{imd}_1^* \otimes \mathfrak{t}^1) \times \mathbb{Z}^{1+b_1(X)}$. As a consequence the integration over \mathcal{G}_* splits into an integral over $\text{imd}_1^* \otimes \mathfrak{t}^1$ and a summation over the components of the vector $\vec{m} = (m_0, \dots, m_{b_1(X)}) \in \mathbb{Z}^{1+b_1(X)}$. For the choice (3.9) one obtains

$$\text{Vol}(\mathcal{G}_*; e^{-\Lambda_{reg}^{\mathcal{G}_*}}) = (\det \Delta_0^{\mathbb{T}_{\beta}^1 \times X} |_{\mathcal{H}^0(\mathbb{T}_{\beta}^1 \times X)^{\perp}})^{-1} \Theta_{b_1(X)}(0 | 2\pi\sqrt{-1} h_{\mathbb{T}_{\beta}^1 \times X}^{(1)}), \quad (3.10)$$

where $\Theta_{b_1(X)}(\cdot | \cdot)$ denotes the $b_1(X)$ -dimensional Riemann-Theta function (A.1).

Now we come to the construction of the functional integral representation for the thermal partition function. Due to the non-trivial bundle structure of the space of thermal gauge fields the separation in gauge independent and gauge dependent degrees of freedom can be done only locally. The idea is to construct integrable measures locally in \mathcal{A} and in the end to paste them together with a partition of unity to obtain a global and integrable measures. This method was originally introduced for studying the relation between stochastic quantization and the conventional Faddeev-Popov quantization scheme [32] and then applied to define an integrable partition function for Yang-Mills theory in order to overcome the Gribov problem [33]. Recently this approach was used for the formulation of an appropriate functional integral representation of generalized p -form gauge fields [39].

Let $\{p_a\}$ be a partition of unity of \mathcal{M}_*^Q subordinate to \mathcal{U}^{A_0} and define $\omega_a^{A_0} := \text{pr}_{\mathcal{G}_*} \circ \varphi_a^{A_0}$, where $\varphi_a^{A_0} : \pi_{\mathcal{A}^Q}^{-1}(U_a) \rightarrow U_a \times \mathcal{G}_*$ is a family of local trivializations of the bundle $\mathcal{A}^Q \xrightarrow{\pi_{\mathcal{A}^Q}} \mathcal{M}_*^Q$ and $\text{pr}_{\mathcal{G}_*} : U_a^{A_0} \times \mathcal{G}_* \rightarrow \mathcal{G}_*$ is the natural projection. Let us introduce the following smooth functional on \mathcal{A}^Q ,

$$\Xi^Q = \sum_a (\pi_{\mathcal{A}^Q}^* p_a) e^{-(\omega_a^{A_0})^* \Lambda_{reg}^{\mathcal{G}_*}}, \quad (3.11)$$

which accounts for the regularized group volume \mathcal{G}_* . Now we define the functional integral representation for the thermal partition function by

$$\mathcal{Z}(\beta, V) = N \sum_{c_1(Q) \in H^2(\mathbb{T}_\beta^1 \times X; \mathbb{Z})} \int_{\mathcal{A}^Q} \frac{vol_{\mathcal{A}^Q}^{\hat{g}}}{Vol(\mathcal{G}; e^{-\Lambda_{reg}^{\hat{g}}})} \Xi^Q e^{-S_{tot}}. \quad (3.12)$$

The partition function is the formal sum over all equivalence classes of principal $U(1)$ bundles over $\mathbb{T}_\beta^1 \times X$ classified by their first Chern-classes $c_1(Q)$ and the functional integration over the corresponding spaces \mathcal{A}^Q of thermal gauge fields. The normalization constant N is temperature- and volume independent and will be fixed later on. The volume of the total gauge group \mathcal{G} is factored out in order to reduce the system to the true physical degrees of freedom. Notice that this is a slight generalization of the original procedure introduced in Ref. [33], where only the volume of the restricted gauge group \mathcal{G}_* was factored out. It will be shown that this proposed partition function is integrable and resolves the Gribov problem. Moreover, the phase e^{-S_θ} gives the weight-factor for the different topological sectors.

Physical observables are regarded as gauge invariant functions on $\mathcal{A}_{\mathbb{T}_\beta^1 \times X}$. The thermal expectation value (TEV) of a physical observable $f \in C^\infty(\mathcal{A}^Q)$ is defined by

$$\langle f \rangle_\beta = \frac{\sum_{c(Q) \in H^2(\mathbb{T}_\beta^1 \times X; \mathbb{Z})} I_\beta^Q(f)}{\mathcal{Z}(\beta, V)}, \quad (3.13)$$

where

$$I_\beta^Q(f) = \int_{\mathcal{A}^Q} \frac{vol_{\mathcal{A}^Q}^{\hat{g}}}{Vol(\mathcal{G}; e^{-\Lambda_{reg}^{\hat{g}}})} e^{-S_{tot}(\beta)} \Xi^Q f. \quad (3.14)$$

As a consequence the TEV of a physical observable is independent of the particular choices of the regularization, the local trivializations and the partition of unity (for an explicit proof see [33]).

3.2 Determination of the free energy

In accordance with the bundle structure of \mathcal{A}^Q , the functional integral in (3.12) can be transformed into an integral over $\mathbb{T}^{1+b_1(X)} \times \mathcal{N} \times \mathcal{G}_*$, which can be explicitly calculated. Using the family of local diffeomorphisms (2.9), one obtains for the transformed measure

$$(\chi_a^{A_0})^* vol_{\mathcal{A}^Q}^{\hat{g}} = \left(\det h_{\mathbb{T}_\beta^1 \times X}^{(1)} \det \Delta_0^{\mathbb{T}_\beta^1 \times X} |_{\mathcal{H}^0(\mathbb{T}_\beta^1 \times X)^\perp} \right)^{\frac{1}{2}} vol_{\mathbb{T}^{1+b_1(X)}}|_{V_a} \wedge vol_{\mathcal{N}} \wedge vol_{\mathcal{G}_*}, \quad (3.15)$$

where $vol_{\mathbb{T}^{1+b_1(X)}}|_{V_a}$ is the induced volume form on $\mathbb{T}^{1+b_1(X)}$, yet restricted to the patch V_a . It follows that $vol_{\mathbb{T}^{1+b_1(X)}} = \hat{i}_0^* vol_{U(1)} \wedge \dots \wedge \hat{i}_{b_1(X)}^* vol_{U(1)}$ where $\hat{i}_k: U(1) \hookrightarrow \mathbb{T}^{1+b_1(X)}$ denotes the k -th inclusion for $k = 0, \dots, b_1(X)$. The restriction of \hat{g} to \mathcal{N} induces a flat metric on that space and an associated volume form $vol_{\mathcal{N}}$. Apart from the conventional Faddeev-Popov determinant of the scalar Laplacian, the factor $\det h_{\mathbb{T}_\beta^1 \times X}^{(1)}$ appears in addition in the induced measure (3.15). This factor carries the topological information of $\mathbb{T}_\beta^1 \times X$ and will be shown to depend on temperature and volume. On the contrary to the zero-temperature case, where this factor as well as the conventional Faddeev-Popov determinant may be neglected, it is essential to keep both factors in the functional integral in the finite temperature context.

A direct calculation gives

$$\begin{aligned}\star_g \left(pr_{\mathbb{T}_\beta^1}^* \varrho^{(1)} \right) &= \frac{1}{\beta} pr_X^* vol_X, \\ \star_g \left(pr_X^* \rho_j^{(1)} \right) &= -pr_{\mathbb{T}_\beta^1}^* vol_{\mathbb{T}_\beta^1} \wedge pr_X^* (\star_\gamma \rho_j^{(1)}),\end{aligned}\tag{3.16}$$

so that the induced metric on $\mathcal{H}^1(\mathbb{T}_\beta^1 \times X)$ admits the following matrix of rank $1 + b_1(X)$,

$$h_{\mathbb{T}_\beta^1 \times X}^{(1)} = \begin{pmatrix} \beta^{-1} V & 0 \\ 0 & \beta h_X^{(1)} \end{pmatrix}.\tag{3.17}$$

Hence

$$\det h_{\mathbb{T}_\beta^1 \times X}^{(1)} = \beta^{b_1(X)-1} V \det h_X^{(1)}.\tag{3.18}$$

Due to the Hodge decomposition theorem it is always possible to choose the fixed background gauge field $A_0 \in \mathcal{A}^Q$ in such a way that the corresponding field strength F_{A_0} becomes harmonic, i.e. $\frac{1}{2\pi\sqrt{-1}} F_{A_0} \in \mathcal{H}_{\mathbb{Z}}^2(\mathbb{T}_\beta^1 \times X)$. Since the cohomology of X is finitely generated, the Chern-class $c_1(Q)$ admits the following (non-canonical) decomposition

$$c_1(Q) = \left(\sum_{i=1}^{b_1(X)} l_i \eta_i^{(1)}, \sum_{j=1}^{b_2(X)} m_j \eta_j^{(2)} + \sum_{k=1}^w y_k t_k^{(2)} \right) \in H^1(X; \mathbb{Z}) \oplus H^2(X; \mathbb{Z}),\tag{3.19}$$

with respect to the Betti bases $(\eta_i^{(1)})_{i=1}^{b_1(X)}$ and $(\eta_j^{(2)})_{j=1}^{b_2(X)}$ for $H^1(X; \mathbb{Z})$ and $H^2(X; \mathbb{Z})$, respectively. Here $b_2(X) = \dim H^2(X; \mathbb{R})$. The torsion subgroup $Tor H^2(X; \mathbb{Z})$ is generated by the basis $(t_k^{(2)})_{k=1}^w$. Hence there exists $r_1, \dots, r_w \in \mathbb{N}$ such that $r_k t_k^{(2)} = 0$ for each $k = 1, \dots, w$. The order of the torsion subgroup is $|Tor H^2(X; \mathbb{Z})| = \prod_{k=1}^w r_k$. Finally the vectors $(l_1, \dots, l_{b_1(X)}) \in \mathbb{Z}^{b_1(X)}$, $(m_1, \dots, m_{b_2(X)}) \in \mathbb{Z}^{b_2(X)}$ and $y_k \in \mathbb{Z}_{r_k} \equiv \mathbb{Z}/r_k \mathbb{Z}$ denote the components with respect to these different generators.

Let $(\rho_j^{(2)})_{j=1}^{b_2(X)}$ be generators for $\mathcal{H}_{\mathbb{Z}}^2(X)$, then $(pr_{\mathbb{T}_\beta^1}^* \varrho^{(1)} \wedge pr_X^* \rho_i^{(1)})_{i=1}^{b_1(X)}$ and $(pr_X^* \rho_j^{(2)})_{j=1}^{b_2(X)}$ are the induced generators for $\mathcal{H}_{\mathbb{Z}}^2(\mathbb{T}_\beta^1 \times X)$. The background field strength can then be expressed in the form

$$F_{A_0} = 2\pi\sqrt{-1} \left[\sum_{i=1}^{b_1(X)} l_i pr_{\mathbb{T}_\beta^1}^* \varrho^{(1)} \wedge pr_X^* \rho_i^{(1)} + \sum_{j=1}^{b_2(X)} m_j pr_X^* \rho_j^{(2)} \right].\tag{3.20}$$

A direct calculation gives

$$\begin{aligned}\star_g \left(pr_X^* \rho_j^{(2)} \right) &= \beta pr_{\mathbb{T}_\beta^1}^* \varrho^{(1)} \wedge pr_X^* (\star_\gamma \rho_j^{(2)}) \\ \star_g \left(pr_{\mathbb{T}_\beta^1}^* \varrho^{(1)} \wedge pr_X^* \rho_i^{(1)} \right) &= \frac{1}{\beta} pr_X^* (\star_\gamma \rho_i^{(1)}),\end{aligned}\tag{3.21}$$

yielding the following metric $(h_{\mathbb{T}_\beta^1 \times X}^{(2)})$ on $\mathcal{H}^2(\mathbb{T}_\beta^1 \times X)$, namely

$$h_{\mathbb{T}_\beta^1 \times X}^{(2)} = \begin{pmatrix} \beta^{-1} h_X^{(1)} & 0 \\ 0 & \beta h_X^{(2)} \end{pmatrix}, \quad (3.22)$$

of rank $b_1(X) + b_2(X)$. Here $(h_X^{(2)})_{ij} = \langle \rho_i^{(2)}, \rho_j^{(2)} \rangle_\gamma$ is the induced metric on $\mathcal{H}^2(X)$.

Let us recall that on an arbitrary compact manifold M the Laplace operator, when restricted to $\mathcal{H}^k(M)^\perp$, splits into the sum $\Delta_k^M|_{\mathcal{H}^k(M)^\perp} = \Delta_k^M|_{\text{imd}_{k-1}} + \Delta_k^M|_{\text{imd}_{k+1}^*}$. Since the spectra of $\Delta_k^M|_{\text{imd}_{k-1}}$ and $\Delta_{k-1}^M|_{\text{imd}_k^*}$ coincide, the spectrum of $\Delta_k^M|_{\mathcal{H}^k(M)^\perp}$ is the union of eigenvalues of $\Delta_k^M|_{\text{imd}_{k+1}^*}$ and of $\Delta_{k-1}^M|_{\text{imd}_k^*}$. In our case $M = \mathbb{T}_\beta^1 \times X$. This implies for the determinant

$$\det \Delta_1^{\mathbb{T}_\beta^1 \times X}|_{\mathcal{H}^1(\mathbb{T}_\beta^1 \times X)^\perp} = (\det \Delta_1^{\mathbb{T}_\beta^1 \times X}|_{\text{imd}_2^*}) (\det \Delta_0^{\mathbb{T}_\beta^1 \times X}|_{\mathcal{H}^0(\mathbb{T}_\beta^1 \times X)^\perp}). \quad (3.23)$$

Using (3.7), (3.15), (3.18), (3.20), (3.22) and (3.23), the integration in (3.14) can be carried out and gives (for $f = 1$)

$$\begin{aligned} I_\beta^Q(1) = & (2\pi)^{(1+b_1(X))} \beta^{\frac{b_1(X)-2}{2}} (\det h_X^{(1)})^{\frac{1}{2}} (\det \Delta_0^{\mathbb{T}_\beta^1 \times X}|_{\mathcal{H}^0(\mathbb{T}_\beta^1 \times X)^\perp}) (\det \Delta_1^{\mathbb{T}_\beta^1 \times X}|_{\mathcal{H}^1(\mathbb{T}_\beta^1 \times X)^\perp})^{-\frac{1}{2}} \\ & \times \exp \left(-\frac{(2\pi)^2}{2} \left[\beta^{-1} \sum_{i,j=1}^{b_1(X)} (h_X^{(1)})_{ij} l_i l_j + \beta \sum_{i,j=1}^{b_2(X)} (h_X^{(2)})_{ij} m_i m_j \right] + 2\pi\sqrt{-1} \sum_{i=1}^{b_1(X)} \theta_i l_i \right) \\ & \times \text{Vol}(U(1); e^{-\Lambda_{reg}^{U(1)}})^{-1} N. \end{aligned} \quad (3.24)$$

The pair of vectors $(\vec{l}, \vec{m}) \in \mathbb{Z}^{b_1(X)} \times \mathbb{Z}^{b_2(X)}$ labels the inequivalent principal $U(1)$ -bundles Q . Using the decomposition (3.19), the sum over the Chern-classes $c_1(Q)$ is performed by summing (3.24) over the free part and the torsion part of $H^2(\mathbb{T}_\beta^1 \times X; \mathbb{Z})$, respectively.

Let us choose $e^{-\Lambda_{reg}^{U(1)}} := (2\pi)^{\frac{b_1(X)}{2}} \beta^{\frac{b_1(X)-2}{2}}$ for the regularizing functional on $U(1)$, then the thermal partition function (3.12) admits the following form

$$\begin{aligned} \mathcal{Z}^\theta(\beta, V) = & (2\pi)^{\frac{b_1(X)}{2}} \beta^{\frac{b_1(X)-2}{2}} (\det h_X^{(1)})^{\frac{1}{2}} (\det \Delta_0^{\mathbb{T}_\beta^1 \times X}|_{\mathcal{H}^0(\mathbb{T}_\beta^1 \times X)^\perp}) (\det \Delta_1^{\mathbb{T}_\beta^1 \times X}|_{\mathcal{H}^1(\mathbb{T}_\beta^1 \times X)^\perp})^{-\frac{1}{2}} \\ & \times \Theta_{b_1(X)} \left(\vec{\theta} | 2\pi\sqrt{-1} \beta^{-1} h_X^{(1)} \right) \Theta_{b_2(X)} \left(0 | 2\pi\sqrt{-1} \beta h_X^{(2)} \right) | \text{Tor} H^2(X; \mathbb{Z}) | N. \end{aligned} \quad (3.25)$$

In order to give the formal determinants arising in (3.25) a mathematical meaning, the zeta-function method will be used as regularization technique. Generally, the regularized determinant of a non-negative, self-adjoint elliptic operator \mathcal{D} of second order on a general manifold M is defined by

$$\det \mathcal{D} = \exp(-\zeta'(0; \mathcal{D})) \equiv \exp \left(-\frac{d}{ds} \Big|_{s=0} \zeta(s; \mathcal{D}) \right). \quad (3.26)$$

Here $\zeta(s; \mathcal{D})$ denotes the zeta-function of the operator \mathcal{D} , defined by

$$\zeta(s; \mathcal{D}) = \sum_{\nu_\alpha(\mathcal{D}) \neq 0} \nu_\alpha(\mathcal{D})^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} \text{Tr}(e^{-t\mathcal{D}} - \Pi^\mathcal{D}), \quad s \in \mathbb{C} \quad (3.27)$$

where the sum runs over the non vanishing eigenvalues $\nu_\alpha(\mathcal{D})$ of \mathcal{D} only. In this sum each eigenvalue appears the same number of times as its multiplicity. This sum is convergent for $\Re(s) > \frac{\dim M}{2}$. The second equation in (3.27) is the heat-kernel representation of the zeta-function based on the Mellin transformation. Therein $\Pi^\mathcal{D}$ is the projector onto the kernel of \mathcal{D} . Let \mathcal{D}' denote the restriction of \mathcal{D} to the non-zero modes, i.e. $\mathcal{D}' := \mathcal{D}|_{\ker \mathcal{D}^\perp}$, then $\zeta(s; \mathcal{D}) = \zeta(s; \mathcal{D}')$ holds by construction. In general, the operator \mathcal{D} possesses the following asymptotic expansion

$$\text{Tr}(e^{-t\mathcal{D}}) \simeq \sum_{k=0}^{\infty} a_k(\mathcal{D}) t^{\frac{k-\dim M}{2}}, \quad \text{for } t \downarrow 0, \quad (3.28)$$

with respect to the asymptotic sequence $t \mapsto t^{\frac{k-\dim M}{2}}$ of functions [40, 41, 42]. The constants $a_k(\mathcal{D})$ are the Seeley coefficients of \mathcal{D} . By using this expansion and splitting the integral in (3.27) into an integral over $[0, 1]$ and $[1, \infty)$ respectively, the zeta-function admits the asymptotic expansion

$$\zeta(s; \mathcal{D}) \simeq \frac{1}{\Gamma(s)} \left(\sum_{\substack{k=0 \\ k \neq \dim M}}^{\infty} \frac{a_k(\mathcal{D})}{s + \frac{k-\dim M}{2}} + \frac{a_{\dim M}(\mathcal{D}) - \dim \ker \mathcal{D}}{s} + r(s; \mathcal{D}) \right), \quad (3.29)$$

where $r(s; \mathcal{D})$ is an analytic function. The zeta function has a meromorphic extension over \mathbb{C} with simple poles at $s_k = \frac{\dim M - k}{2}$ for $k \in \mathbb{N}_0$ and residue $\text{Res}_{s=s_k} [\zeta(s; \mathcal{D})] = \frac{a_k(\mathcal{D})}{\Gamma(\frac{\dim M - k}{2})}$ at $s = s_k$. Since $\lim_{s \rightarrow 0} s\Gamma(s) = 1$, the zeta function is analytic in the origin and one finds that $\zeta(0; \mathcal{D}) = a_{\dim M}(\mathcal{D}) - \dim \ker \mathcal{D}$. In our case this leads to

$$\zeta(0; \Delta_p^{\mathbb{T}_\beta^1 \times X}) \equiv \zeta(0; \Delta_p^{\mathbb{T}_\beta^1 \times X}|_{\mathcal{H}^p(\mathbb{T}_\beta^1 \times X)^\perp}) = a_{n+1}(\Delta_p^{\mathbb{T}_\beta^1 \times X}) - b_p(\mathbb{T}_\beta^1 \times X) \quad \text{for } p = 0, 1. \quad (3.30)$$

Let us denote the eigenvalues of Δ_s^X by $\nu_{l_s}(\Delta_s^X)$, where l_s runs over an appropriate subset $J^{(s)} \subset \mathbb{Z}$. Notice that the eigenvalues of $\Delta_r^{\mathbb{T}_\beta^1}$ ($r = 0, 1$) are given by $\nu_k(\Delta_r^{\mathbb{T}_\beta^1}) = (\frac{2\pi k}{\beta})^2$ with $k \in \mathbb{Z}$. Then the spectrum of $\Delta_p^{\mathbb{T}_\beta^1 \times X}|_{\mathcal{H}^p(\mathbb{T}_\beta^1 \times X)^\perp}$ ($p = 0, 1$) contains the following set of non-vanishing real numbers

$$\begin{aligned} \text{Spec}(\Delta_p^{\mathbb{T}_\beta^1 \times X}|_{\mathcal{H}^p(\mathbb{T}_\beta^1 \times X)^\perp}) &= \\ &= \{\nu_{(k, l_s)}^{(r, s)} := \nu_k(\Delta_r^{\mathbb{T}_\beta^1}) + \nu_{l_s}(\Delta_s^X) \neq 0 | r + s = p, (k, l_s) \in \mathbb{Z} \times J^{(s)}\}. \end{aligned} \quad (3.31)$$

In terms of these eigenvalues, the corresponding zeta functions can be written as convergent infinite series (for $\Re(s) > \frac{n+1}{2}$)

$$\zeta(s; \Delta_p^{\mathbb{T}_\beta^1 \times X} |_{\mathcal{H}^p(\mathbb{T}_\beta^1 \times X)^\perp}) = \begin{cases} \sum_{(k, l_0) \in \mathbb{Z} \times J^{(0)}} \left[\nu_{(k, l_0)}^{(0,0)} \right]^{-s}, & \text{if } p = 0, \\ \sum_{(k, l_0) \in \mathbb{Z} \times J^{(0)}} \left[\nu_{(k, l_0)}^{(1,0)} \right]^{-s} + \sum_{(k, l_1) \in \mathbb{Z} \times J^{(1)}} \left[\nu_{(k, l_1)}^{(0,1)} \right]^{-s}, & \text{if } p = 1, \end{cases} \quad (3.32)$$

where each eigenvalue appears as often as its multiplicity. Remark that $\nu_{(k, l_0)}^{(0,0)} = \nu_{(k, l_0)}^{(1,0)}$.

We notice that there is an intrinsic ambiguity in the definition of the zeta function (3.27) due to the fact that the eigenvalues of the Laplace operators are not dimensionless. As a consequence the partition function would admit a dimension as well. To restore this, an appropriate scale factor μ of mass dimension $[\mu] = 1$ has to be introduced [34]. Instead of (3.27) one introduces therefore the dimensionless zeta function $\zeta_\mu(s; \mathcal{D}) := \sum_{\nu_\alpha(\mathcal{D}) \neq 0} (\mu^{-2} \nu_\alpha(\mathcal{D}))^{-s}$. Formally, this gives $\zeta_\mu(s; \mathcal{D}) = \mu^{2s} \zeta(s; \mathcal{D})$. In the following all determinants appearing in (3.25) are expressed in terms of these scale dependent but dimensionless zeta functions, i.e.

$$\det_\mu(\Delta_p^{\mathbb{T}_\beta^1 \times X} |_{\mathcal{H}^p(\mathbb{T}_\beta^1 \times X)^\perp}) := \exp \left[-\frac{d}{ds} \Big|_{s=0} \zeta_\mu(s; \Delta_p^{\mathbb{T}_\beta^1 \times X} |_{\mathcal{H}^p(\mathbb{T}_\beta^1 \times X)^\perp}) \right], \quad p = 0, 1. \quad (3.33)$$

However, even with this substitution the partition function (3.25) is not dimensionless. This traces back to the occurrence of the two Jacobian determinants (3.7) and (3.18), whose dimensions must be corrected accordingly. In our convention the coupling constant is set to 1, which leads to the mass dimensions $[A] = \frac{n-1}{2}$ and $[F_A] = \frac{n+1}{2}$ for the components of A and F_A , respectively. Consequently, one obtains the mass dimensions $[h_X^{(1)}] = -1$ and $[h_X^{(2)}] = 1$.

In order to get a dimensionless partition function we propose the following replacements in (3.7) and (3.17), namely

$$\begin{aligned} \beta V &\mapsto \beta V \mu^{n+1} \\ h_{\mathbb{T}_\beta^1 \times X}^{(1)} &\mapsto \text{diag} \left(\beta^{-1} V \mu^{n-1}, \beta h_X^{(1)} \mu^2 \right). \end{aligned} \quad (3.34)$$

Alternatively, this substitution could be implemented formally by choosing the normalization constant $N = \mu^{b_1(X)-1}$ in (3.25).

As a result, the free energy $\mathcal{F}^\theta(\beta; V) = -\frac{1}{\beta} \ln \mathcal{Z}^\theta(\beta; V)$ admits now the form

$$\begin{aligned} \mathcal{F}^\theta(\beta; V) &= \frac{2 - b_1(X)}{2\beta} \ln \beta - \frac{1}{2\beta} \ln \det h_X^{(1)} + \frac{1}{\beta} \zeta'(0; \Delta_0^{\mathbb{T}_\beta^1 \times X} |_{\mathcal{H}^0(\mathbb{T}_\beta^1 \times X)^\perp}) \\ &\quad - \frac{1}{2\beta} \zeta'(0; \Delta_1^{\mathbb{T}_\beta^1 \times X} |_{\mathcal{H}^1(\mathbb{T}_\beta^1 \times X)^\perp}) - \frac{1}{\beta} \ln \Theta_{b_1(X)} \left(\bar{\theta} | 2\pi \sqrt{-1} \beta^{-1} h_X^{(1)} \right) \\ &\quad - \frac{1}{\beta} \ln \Theta_{b_2(X)} \left(0 | 2\pi \sqrt{-1} \beta h_X^{(2)} \right) - \frac{b_1(X)}{2\beta} \ln 2\pi - \frac{1}{\beta} \ln |Tor H^2(X; \mathbb{Z})| \\ &\quad + \left(\frac{1}{\beta} \zeta(0; \Delta_0^{\mathbb{T}_\beta^1 \times X} |_{\mathcal{H}^0(\mathbb{T}_\beta^1 \times X)^\perp}) - \frac{1}{2\beta} \zeta(0; \Delta_1^{\mathbb{T}_\beta^1 \times X} |_{\mathcal{H}^1(\mathbb{T}_\beta^1 \times X)^\perp}) + \frac{1 - b_1(X)}{2\beta} \right) \ln \mu^2. \end{aligned} \quad (3.35)$$

In the next step we want to express the free energy in terms of the geometry of the spatial manifold X : Given any non-negative, self-adjoint elliptic operator D of second order on X with spectrum $\{\nu_\alpha(D) | \alpha \in J\}$, we define the following two auxiliary quantities

$$\begin{aligned}
\mathcal{I}(s; D) &:= \sum_{(k, \alpha) \in (\mathbb{Z} \times J)'} \left[\left(\frac{2\pi k}{\beta} \right)^2 + \nu_\alpha(D) \right]^{-s} \\
\hat{\mathcal{I}}(s; D) &:= \sum_{k \in \mathbb{Z}} \sum_{\alpha \in J'} \left[\left(\frac{2\pi k}{\beta} \right)^2 + \nu_\alpha(D') \right]^{-s}.
\end{aligned} \tag{3.36}$$

The prime in the first formula indicates that the sum runs only over those indices (k, α) such that $(\frac{2\pi k}{\beta})^2 + \nu_\alpha(D) \neq 0$. In the second line the index set $J' \subseteq J$ labels the eigenvalues of the restricted operator D' . One immediately finds that

$$\mathcal{I}(s; D) = \hat{\mathcal{I}}(s; D) + 2(\dim \ker D) \left(\frac{\beta}{2\pi} \right)^{2s} \zeta_R(2s), \tag{3.37}$$

where $\zeta_R(s)$ is the Riemann zeta function. Hence (3.32) can be rewritten in the form

$$\zeta(s; \Delta_p^{\mathbb{T}_\beta^1 \times X} |_{\mathcal{H}^p(\mathbb{T}_\beta^1 \times X)^\perp}) = \begin{cases} \mathcal{I}(s; \Delta_0^X) & \text{if } p = 0, \\ \mathcal{I}(s; \Delta_0^X) + \mathcal{I}(s; \Delta_1^X) & \text{if } p = 1. \end{cases} \tag{3.38}$$

Applying the Mellin transform to $\hat{\mathcal{I}}(s; D)$ and performing the integration over t give

$$\begin{aligned}
\hat{\mathcal{I}}(s; D) &= \frac{1}{\Gamma(s)} \int_0^\infty dt \, t^{s-1} \sum_{k \in \mathbb{Z}} e^{-(\frac{2\pi}{\beta})^2 k^2 t} \sum_{\alpha \in J'} e^{-\nu_\alpha(D')t} \\
&= \frac{\beta}{2\sqrt{\pi}\Gamma(s)} \int_0^\infty dt \, t^{s-\frac{3}{2}} \sum_{\alpha \in J'} e^{-\nu_\alpha(D')t} + \frac{\beta}{\sqrt{\pi}\Gamma(s)} \int_0^\infty dt \, t^{s-\frac{3}{2}} \sum_{k=1}^\infty e^{-(\frac{k^2 \beta^2}{4t})} \sum_{\alpha \in J'} e^{-\nu_\alpha(D')t} \\
&= \frac{\beta}{2\sqrt{\pi}} \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \zeta(s-\frac{1}{2}; D) + \frac{2^{-s-\frac{3}{2}} \beta^{s+\frac{1}{2}}}{\sqrt{\pi}\Gamma(s)} \sum_{k \in \mathbb{Z}} \sum_{\alpha \in J'} \left[\frac{k}{\sqrt{\nu_\alpha(D')}} \right]^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(k\beta\sqrt{\nu_\alpha(D')}).
\end{aligned} \tag{3.39}$$

Here $K_\nu(s)$ is the modified Bessel function of the second kind [43]. In order to calculate the derivative of $\mathcal{I}'(0; D)$ in $s = 0$, we notice that $\zeta(s; D)$ admits a Laurent series expansion in $s = -\frac{1}{2}$, namely

$$\zeta(s - \frac{1}{2}; D) = \frac{\text{Res}_{s=-\frac{1}{2}} [\zeta(s; D)]}{s} + FP_{s=-\frac{1}{2}} [\zeta(s; D)] + \sum_{k=1}^\infty \tilde{\sigma}_k s^k, \tag{3.40}$$

where FP denotes the finite part of the zeta function in $s = -\frac{1}{2}$. Recall that for an arbitrary meromorphic function f , the finite part in $s_0 \in \mathbb{C}$ is defined by

$$FP_{s=s_0}[f] := \lim_{\epsilon \rightarrow 0^+} \frac{1}{2\pi\sqrt{-1}} \oint_{|s-s_0|=\epsilon} \frac{f(s)}{s-s_0} ds. \tag{3.41}$$

(see e.g. [44] for the definition and properties). For the zeta function of D this implies

$$FP_{s=s_0}[\zeta(s; D)] = \lim_{\epsilon \rightarrow 0} \frac{1}{2} \left[\zeta(-\frac{1}{2} - \epsilon; D) + \zeta(-\frac{1}{2} + \epsilon; D) \right]. \tag{3.42}$$

Taking the expansions $\frac{1}{\Gamma(s)} = s + \gamma s^2 + \mathcal{O}(s^3)$ and $\Gamma(s - \frac{1}{2}) = \Gamma(-\frac{1}{2})(1 + \Psi(-\frac{1}{2})s + \mathcal{O}(s^2))$ for small s , where γ is the Euler-Mascheroni number and Ψ is the Digamma function [43], and using that the relation $\lim_{s \rightarrow 0} \frac{d}{ds}(\frac{h(s)}{\Gamma(s)}) = h(0)$ holds for any regular function h , one finally obtains

$$\begin{aligned} \mathcal{I}'(0; D) = & -2(\dim \ker D) \ln \beta - 2 \sum_{\alpha \in J'} \ln \left[1 - e^{-\beta \sqrt{\nu_\alpha(D')}} \right] \\ & - \beta \left(FP_{s=-\frac{1}{2}} [\zeta(s; D)] + 2(1 - \ln 2) Res_{s=-\frac{1}{2}} [\zeta(s; D)] \right). \end{aligned} \quad (3.43)$$

Using the duality formula (A.3) the heat kernel expansion of $\Delta_p^{\mathbb{T}_\beta^1}$ becomes

$$Tr(e^{-t\Delta_p^{\mathbb{T}_\beta^1}}) = \Theta_1(0|\sqrt{-1} \frac{4\pi t}{\beta^2}) = \frac{\beta}{2\sqrt{\pi t}} \Theta_1(0|\sqrt{-1} \frac{\beta^2}{4\pi t}) \simeq \frac{\beta}{2\sqrt{\pi t}} + \mathcal{O}(e^{-\frac{1}{t}}), \quad \text{for } t \downarrow 0, \quad (3.44)$$

for $p = 0, 1$. This implies $a_k(\Delta_p^{\mathbb{T}_\beta^1}) = \frac{\beta}{2\sqrt{\pi}} \delta_{k,0}$. By comparing the asymptotic expansions of $\Delta_p^{\mathbb{T}_\beta^1 \times X}$ and Δ_p^X using (3.31), one finds the following relation between the corresponding Seeley coefficients,

$$\begin{aligned} a_k(\Delta_0^{\mathbb{T}_\beta^1 \times X}) &= \frac{\beta}{2\sqrt{\pi}} a_k(\Delta_0^X) \\ a_k(\Delta_1^{\mathbb{T}_\beta^1 \times X}) &= \frac{\beta}{2\sqrt{\pi}} [a_k(\Delta_1^X) + a_k(\Delta_0^X)]. \end{aligned} \quad (3.45)$$

We want to calculate the sum in the second term of (3.43) in the case of $D = \Delta_1^X$. Let $\{\nu_{\alpha_1}(\Delta_1^X|_{imd_2^*}) | \alpha_1 \in J_1\}$ and $\{\nu_{\alpha_0}(\Delta_0^X|_{imd_1^*}) | \alpha_0 \in J_0\}$ be the spectra of $\Delta_1^X|_{imd_2^*}$ and $\Delta_0^X|_{imd_1^*}$, respectively. Since the spectrum of $\Delta_p^X|_{\mathcal{H}^p(X)^\perp}$ is the union of eigenvalues of $\Delta_p^X|_{imd_{p+1}^*}$ and $\Delta_{p-1}^X|_{imd_p^*}$ the sum runs over $\nu_{\alpha_1}(\Delta_1^X|_{imd_2^*})$ and $\nu_{\alpha_0}(\Delta_0^X|_{imd_1^*})$, respectively. When inserting the explicit expression (3.38) into (3.35) only the sum over the eigenvalues of $\Delta_1^X|_{imd_2^*}$ survives. Furthermore, this also implies $\zeta(s; \Delta_1^X|_{imd_2^*}) = \zeta(s; \Delta_1^X) - \zeta(s; \Delta_0^X)$. In terms of the Seeley coefficients, the residue of $\zeta(s; D)$ in $s = -\frac{1}{2}$ is given by

$$Res_{s=-\frac{1}{2}} [\zeta(s; D)] = -\frac{a_{\dim X+1}(D)}{2\sqrt{\pi}}. \quad (3.46)$$

Due to the linear structure of FP , Res and applying the duality formula (A.3) once again to the $b_1(X)$ -dimensional Riemann Theta function, we finally get the functional integral representation for the free energy

$$\begin{aligned} \mathcal{F}^\theta(\beta; V) = & \frac{1}{2} FP_{s=-\frac{1}{2}} [\zeta(s; \Delta_1^X|_{imd_2^*})] + \frac{1}{2} Res_{s=-\frac{1}{2}} [\zeta(s; \Delta_1^X|_{imd_2^*})] \ln \left(\frac{e\mu}{2} \right)^2 \\ & + \frac{1}{\beta} \sum_{\alpha_1 \in J_1} \ln \left[1 - e^{-\beta \sqrt{\nu_{\alpha_1}(\Delta_1^X|_{imd_2^*})}} \right] - \frac{1}{\beta} \ln \Theta_{b_1(X)} \left[\begin{matrix} \vec{\theta} \\ 0 \end{matrix} \right] \left(0 \left| \frac{\sqrt{-1}}{2\pi} \beta (h_X^{(1)})^{-1} \right. \right) \\ & - \frac{1}{\beta} \ln \Theta_{b_2(X)} \left(0 \left| 2\pi \sqrt{-1} \beta (h_X^{(2)}) \right. \right) - \frac{1}{\beta} \ln |Tor H^2(X; \mathbb{Z})|. \end{aligned} \quad (3.47)$$

It will be shown in the next section that the Hamiltonian formalism gives exactly the same result. Eq. (3.47) shows explicitly that the ambiguity in the free energy is independent of the temperature.

Before closing this section we want to annotate briefly why the conventional Faddeev-Popov procedure is not applicable in the present case (For a detailed account refer to [29]). In the Faddeev-Popov approach the starting point would be the functional integral (3.12), however with the choices $\Lambda_{reg}^{\mathcal{G}} = \Lambda_{reg}^{\mathcal{G}^*} = 0$. If the gauge fields were constrained by the covariant gauge condition $d^*(A - A_0) = 0$, this would lead to the well known additional gauge fixing term $\frac{1}{2}\|d_1^*(A - A_0)\|^2$ in the action. This term in combination with the classical Maxwell action would then give the unrestricted kinetic Laplace operator $\Delta_1^{\mathbb{T}_\beta^1 \times X}$ acting on the thermal gauge fields in the total action functional. Since $\mathcal{H}^1(\mathbb{T}_\beta^1 \times X) \neq 0$ this operator is not invertible, so that the functional integral over the gauge fields would diverge. This is precisely the Gribov problem in the abelian case. Even if this problem was solved ex post by restricting the Laplace operator to the non-harmonic forms, the Faddeev-Popov determinant related to the covariant gauge condition would be - following the conventional approach - only the factor $\det \Delta_0^{\mathbb{T}_\beta^1 \times X} |_{\mathcal{H}^0(\mathbb{T}_\beta^1 \times X)}$.

In our approach there is an additional multiplicative factor $\det h_{\mathbb{T}_\beta^1 \times X}^{(1)}$ which turns out to be essential in the finite temperature context and which appears naturally as part of the Jacobian of the transformation (2.9). It gives rise to the first term on the right hand side of (3.35). Its first part, namely $\frac{1}{\beta} \ln \beta$, is related to $\mathcal{H}^1(\mathbb{T}_\beta^1)$ and is present even if $b_1(X) = 0$. However, this term is compensated by the zero-mode subtraction coming from the expansion (3.37) of the zeta function of the Laplace operator on $\mathbb{T}_\beta^1 \times X$ in terms of the zeta function associated to the corresponding Laplace operators on X . Without this cancellation, the entropy (5.1) would have admitted a logarithmic divergence in the zero-temperature limit. In this context we would like to refer to the scalar case, where this logarithmic term was present in [17] but disappeared in [23] due to a modification of the functional integral formula for the free energy. By this modification the zero modes of the scalar Laplacian are taken into account correctly and the equality with the operator (Hamiltonian) approach is provided.

The second contribution, namely $\frac{b_1(X)}{2\beta} \ln \beta$ (i.e. the second term in (3.35)) is completely absorbed in the Riemann Theta function due to its duality property. Moreover, our treatment of the Gribov problem automatically rules out the zero modes of the kinetic Laplace operator right from the beginning, so that the functional integral can be carried out, yet giving a finite result.

4 The free energy in the Hamiltonian approach

In this section the quantization of Maxwell theory is studied from the canonical (Hamiltonian) point of view. The aim is to determine the corresponding Hamilton operator \hat{H}^θ in the presence of θ -vacua and to calculate the (thermal) partition function and the free energy according to

$$\begin{aligned} \hat{\mathcal{Z}}^\theta(\beta, V) &= Tr \left(e^{-\beta \hat{H}^\theta} \right) \\ \hat{\mathcal{F}}^\theta(\beta, V) &= -\frac{1}{\beta} \ln \hat{\mathcal{Z}}^\theta(\beta, V). \end{aligned} \tag{4.1}$$

The trace is taken along the physical (i.e. gauge invariant) states. If the theory possesses different topological sectors (like it is in our case), one has to perform in addition a sum over these sectors as well. We will return to this topic below. In a first step the objects in (4.1) are marked with a caret in order to distinguish them from the partition function and free energy obtained in the functional integral formalism.

As in the previous section, let Q be a principal $U(1)$ -bundle over $\mathbb{T}_\beta^1 \times X$ and consider the pull-back bundle $\underline{Q} := \hat{i}_X^* Q$ over X , where $\hat{i}_X: X \hookrightarrow \mathbb{T}_\beta^1 \times X$ is the canonical inclusion $\hat{i}_X(x) := (1, x)$. For the fixed-time canonical formalism we consider - as usual - the principal $U(1)$ -bundle $\mathbb{R} \times \underline{Q}$ over the space-time manifold $\mathbb{R} \times X$.

Let \mathcal{A}^Q denote the space of connections of \underline{Q} and let $\mathcal{A}^s = \Omega^0(X; \mathfrak{t}^1)$ be the space of scalar gauge potentials. The tangent bundle $T(\mathcal{A}^Q \times \mathcal{A}^s) \cong \mathcal{A}^Q \times \mathcal{A}^s \times \Omega^1(X; \mathfrak{t}^1) \times \Omega^0(X; \mathfrak{t}^1)$ is the corresponding configuration space, which is parametrized by the coordinates $(\underline{A}, \underline{A}^s, \underline{\dot{A}}, \underline{\dot{A}}^s)$. After performing the analytic continuation, the topological action S_θ (3.2) becomes real. The Lagrangian associated to the classical Maxwell action S_{tot} admits the form

$$L^\theta(\underline{A}, \underline{A}^s, \underline{\dot{A}}, \underline{\dot{A}}^s) = \frac{1}{2} \|\underline{\dot{A}} - d_0 \underline{A}^s\|_\gamma^2 - \frac{1}{2} \|F_{\underline{A}}\|_\gamma^2 + \frac{1}{2\pi} \hat{\gamma}(\underline{\dot{A}} - d_0 \underline{A}^s, \underline{\theta}), \quad (4.2)$$

where $F_{\underline{A}} = d_1 \underline{A}$ is the magnetic field. Here $\|\cdot\|_\gamma$ refers to the norm which is induced by the metric $\hat{\gamma}(v_1, v_2) := - \int_X v_1 \wedge \star_\gamma v_2$ on \mathcal{A}^Q , where $v_1, v_2 \in \Omega^1(X; \mathfrak{t}^1)$. Since $\hat{\gamma}(\underline{\dot{A}} - d_0 \underline{A}^s, \underline{\theta}) = \hat{\gamma}(\underline{\dot{A}}, \underline{\theta})$ holds, the topological term is a total time derivative, so that the classical equations of motion remain unchanged despite the addition of this particular term. By performing the Legendre transformation one gets the corresponding Hamiltonian in the phase space $T^*(\mathcal{A}^Q \times \mathcal{A}^s)$ with conjugate momenta

$$\underline{\Pi} = \frac{\delta L^\theta}{\delta \underline{\dot{A}}} = \underline{\dot{A}} - d \underline{A}^s + \frac{\underline{\theta}}{2\pi}, \quad \underline{\Pi}^s = \frac{\delta L^\theta}{\delta \underline{\dot{A}}^s} = 0. \quad (4.3)$$

The phase space $T^*(\mathcal{A}^Q \times \mathcal{A}^s)$ is equipped with the canonical symplectic form. Let $\underline{A}_0 \in \mathcal{A}^Q$ be a fixed background connection satisfying $d_2^* F_{\underline{A}_0} = 0$. The non-vanishing Poisson brackets for the basic linear phase space functionals $\underline{A}_u := \hat{\gamma}(\underline{A} - \underline{A}_0, u)$, $\underline{A}_u^s := \hat{\gamma}(\underline{A}^s, v)$, $\underline{\Pi}_{u'} := \hat{\gamma}(\underline{\Pi}, u')$ and $\underline{\Pi}_{v'}^s := \hat{\gamma}(\underline{\Pi}^s, v')$ are given by

$$\{\underline{A}_u, \underline{\Pi}_{u'}\} = \hat{\gamma}(u, u'), \quad \{\underline{A}_v^s, \underline{\Pi}_{v'}^s\} = \hat{\gamma}(v, v'), \quad (4.4)$$

where $u, u' \in \Omega^1(X; \mathfrak{t}^1)$ and $v, v' \in \Omega^0(X; \mathfrak{t}^1)$. The Lagrangian L^θ is singular and leads to two first class constraints in the phase space, namely $\underline{\Pi}^s = 0$ and the Gauss law $d^* \underline{\Pi} = 0$. By choosing the temporal gauge $\underline{A}^s = 0$, the first constraint is solved. The dynamical system is restricted to the submanifold $\mathcal{C} := \{(\underline{A}, \underline{\Pi}) \in T^* \mathcal{A}^Q | d^* \underline{\Pi} = 0\}$ and governed by the induced Hamiltonian

$$H^\theta(\underline{A}, \underline{\Pi}) = \frac{1}{2} \|\underline{\Pi} - \frac{\underline{\theta}}{2\pi}\|_\gamma^2 + \frac{1}{2} \|F_{\underline{A}}\|_\gamma^2. \quad (4.5)$$

The corresponding group of gauge symmetries is $\mathcal{G}^Q = C^\infty(X; U(1))$. Let $\mathcal{G}_*^Q := \mathcal{G}^Q / U(1)$ be the restricted gauge group, which acts freely on \mathcal{A}^Q . In fact, the Gauss law constraint is related to the symmetry under the identity component $\mathcal{G}_{*,0}^Q$ of \mathcal{G}_*^Q and can be solved by factoring out this subgroup. This leads to the quotient space $\mathcal{C} / \mathcal{G}_{*,0}^Q$. However, since \mathcal{G}_*^Q is not connected, i.e. $\pi_0(\mathcal{G}_*^Q) \cong \mathbb{Z}^{b_1(X)}$, this is not the true physical phase space \mathcal{P} . That space is obtained by

taking out also the large gauge transformations. As a result $\mathcal{P} = (\mathcal{C}/\mathcal{G}_{*,0}^Q)/\pi_0(\mathcal{G}_*^Q) = \mathcal{C}/\mathcal{G}_*^Q$, which is symplectically equivalent to the cotangent bundle $T^*\mathcal{M}_*^Q$ of the space of gauge orbits $\mathcal{M}_*^Q = \mathcal{A}^Q/\mathcal{G}_*^Q$. At first sight this seems to be not the final result, since only the restricted gauge group \mathcal{G}_*^Q instead of \mathcal{G}^Q has been considered so far. But since $\mathcal{A}^Q/\mathcal{G}^Q \cong \mathcal{M}_*^Q$, $T^*\mathcal{M}_*^Q$ can be really viewed as the true physical phase space \mathcal{P} of the classical system.

In the Hamiltonian approach the underlying geometrical structure is the principal \mathcal{G}_*^Q -bundle $\mathcal{A}^Q \rightarrow \mathcal{M}_*^Q$. In order to obtain an explicit parametrization of \mathcal{M}_*^Q , we can apply propositions 1 and 2 of section 2 (by replacing the base manifold $\mathbb{T}_\beta^1 \times X$ with X in the assumptions). Thus \mathcal{M}_*^Q admits the structure of a trivializable vector bundle over $\mathbb{T}^{b_1(X)}$ with typical fiber $\underline{\mathcal{N}} := \text{imd}_2^* \otimes \mathfrak{t}^1$, where $d_2^*: \Omega^2(X; \mathfrak{t}^1) \rightarrow \Omega^1(X; \mathfrak{t}^1)$. The trivialization induces a vector bundle isomorphism $\phi: T^*(\mathbb{T}^{b_1(X)} \times \underline{\mathcal{N}}) \rightarrow T^*\mathcal{M}_*^Q$ between the corresponding cotangent bundles. If $(z_1, \dots, z_{b_1(X)}; p_1, \dots, p_{b_1(X)})$ are coordinates for $T^*\mathbb{T}^{b_1(X)}$ and (τ, Υ) are those for $T^*\underline{\mathcal{N}}$, then ϕ is given by

$$\begin{aligned} \phi(z_1, \dots, z_{b_1(X)}, \tau; p_1, \dots, p_{b_1(X)}, \Upsilon) = \\ = \left(\left[\underline{A}_0 + 2\pi\sqrt{-1} \sum_{j=1}^{b_1(X)} \sigma_{a_j}(z_j) \rho_j^{(1)} + \tau \right], \frac{\sqrt{-1}}{2\pi} \sum_{j,k=1}^{b_1(X)} (h_X^{(1)})_{jk}^{-1} p_j \rho_k^{(1)} + \Upsilon \right), \end{aligned} \quad (4.6)$$

where the brackets denote the equivalence class in \mathcal{M}_*^Q and $s_{a_j}: V_{a_j} \subset \mathbb{T}^1 \rightarrow \mathbb{R}$ are the local sections of the universal covering $\mathbb{R} \rightarrow \mathbb{T}^1$ (as defined in Section 2). It can be easily shown that ϕ is globally well defined. Notice that we have parametrized $T^*\mathcal{M}_*^Q$ by pairs $([\underline{A}], \underline{\Pi})$ satisfying $d_1^* \underline{\Pi} = 0$. The inverse map reads

$$\begin{aligned} \phi^{-1}([\underline{A}], \underline{\Pi}) = & \left(e^{\int_X (\underline{A} - \underline{A}_0) \wedge \rho_1^{(n-1)}}, \dots, e^{\int_X (\underline{A} - \underline{A}_0) \wedge \rho_{b_1(X)}^{(n-1)}}, d_2^* G_2(F_{\underline{A}} - F_{\underline{A}_0}), \right. \\ & \frac{2\pi}{\sqrt{-1}} \sum_{k=1}^{b_1(X)} (h_X^{(1)})_{1k} \int_X \underline{\Pi} \wedge \rho_k^{(n-1)}, \dots, \frac{2\pi}{\sqrt{-1}} \sum_{k=1}^{b_1(X)} (h_X^{(1)})_{b_1(X)k} \int_X \underline{\Pi} \wedge \rho_k^{(n-1)}, \\ & \left. d_2^* G_2 d_1 \underline{\Pi} \right). \end{aligned} \quad (4.7)$$

The field strength $F_{\underline{A}_0} = d\underline{A}_0$ of the background gauge potential classifies the topologically non-trivial monopole configurations. With respect to the chosen Betti-basis (see section 2) one gets

$$F_{\underline{A}_0} = 2\pi\sqrt{-1} \sum_{k=1}^{b_2(X)} m_k \rho_k^{(2)}, \quad m_k \in \mathbb{Z}. \quad (4.8)$$

In these adapted coordinates the classical Hamiltonian splits into two independent dynamical subsystems with the phases spaces $T^*\mathbb{T}^{b_1(X)}$ and $T^*\underline{\mathcal{N}}$, respectively. The dynamics is governed by the Hamiltonian $\phi^* H^\theta = H_{\text{harm}}^\theta + H_{\text{trans}}$, where

$$\begin{aligned}
H_{harm}^\theta(z_1, \dots, z_{b_1(X)}; p_1, \dots, p_{b_1(X)}) &= \\
&= \frac{1}{2(2\pi)^2} \sum_{i,j=1}^{b_1(X)} (h_X^{(1)})_{ij}^{-1} (p_i - 2\pi\theta_i)(p_j - 2\pi\theta_j) + \frac{(2\pi)^2}{2} \sum_{k,l=1}^{b_2(X)} (h_X^{(2)})_{kl} m_k m_l. \quad (4.9)
\end{aligned}$$

and

$$H_{trans}(\tau; \Upsilon) = \frac{1}{2} \|\Upsilon\|_\gamma^2 + \frac{1}{2} \hat{\gamma}(\tau, \Delta_1|_{\underline{N}} \tau). \quad (4.10)$$

In order to quantize these two subsystems we determine the Poisson brackets between the conjugate variables. Let us define the angles $q_j = \frac{1}{2\pi\sqrt{-1}} \int_X (\underline{A} - \underline{A}_0) \wedge \rho_j^{(n-1)}$ for $j = 1, \dots, b_1(X)$, which can be regarded as coordinates of $\mathbb{R}^{b_1(X)}$ (i.e. of the universal cover of $\mathbb{T}^{b_1(X)}$). By definition $z_j = e^{2\pi\sqrt{-1}q_j}$ and from (4.4) we find $\{q_j, p_k\} = \delta_{jk}$. Each gauge transformation $v \in \mathcal{G}^Q$ of the gauge fields, i.e. $\underline{A} \mapsto \underline{A}^v$, induces a translation $q_j \mapsto q_j + \alpha_j$, where $\alpha_j = \frac{1}{2\pi\sqrt{-1}} \int_{c_j} v^* \vartheta^{U(1)} \in \mathbb{Z}$ are the winding numbers related to v and $j = 1, \dots, b_1(X)$. The Poisson bracket of the linear phase space functionals $\tau_u := \hat{\gamma}(\tau, u)$ and $\Upsilon_{u'} := \hat{\gamma}(\Upsilon, u')$ yields $\{\tau_u, \Upsilon_{u'}\} = \hat{\gamma}(u, d_2^* G_2 d_1 u')$, where $u, u' \in \Omega^1(X; \mathfrak{t}^1)$.

In the Schrödinger representation the Hilbert space \mathfrak{H} of physical states splits into the tensor product $\mathfrak{H} = \mathfrak{H}_{harm} \otimes \mathfrak{H}_{trans}$ of Hilbert spaces of sections of line bundles over $\mathbb{T}^{b_1(X)}$ and over \underline{N} , respectively. The \mathcal{G}^Q -invariant metric $\hat{\gamma}$ induces a natural connection in the bundle $\mathcal{A}^Q \rightarrow \mathcal{M}_*^Q$ by declaring the orthogonal complement to the fibers as horizontal subbundle. The metric restricted to this subbundle finally induces a metric $\hat{\gamma}$ on \mathcal{M}_*^Q . With respect to the diffeomorphism $\mathcal{M}_*^Q \cong \mathbb{T}^{b_1(X)} \times \underline{N}$ this metric splits into the direct sum $\hat{\gamma} = h_X^{(1)} \oplus 1_{\underline{N}}$, where $1_{\underline{N}}$ is the induced flat metric on \underline{N} . Let $vol_{\underline{N}}$ denote the induced volume form, then the (formal) inner product in \mathfrak{H} is given by

$$\langle \Psi_1, \Psi_2 \rangle_{\mathfrak{H}} = \int_{\mathbb{T}^{b_1(X)}} vol_{\mathbb{T}^{b_1(X)}} (\det h_X^{(1)})^{\frac{1}{2}} \bar{\psi}_1 \bar{\psi}_2 \int_{\underline{N}} vol_{\underline{N}} \varphi_1 \bar{\varphi}_2, \quad \Psi_i = \psi_i \otimes \varphi_i, \quad i = 1, 2. \quad (4.11)$$

The Hamilton operator \hat{H}_{harm}^θ is obtained from (4.9) by substituting the classical momenta by the operators $\hat{p}_i = \frac{1}{\sqrt{-1}} \frac{\partial}{\partial q_i}$. This leads to

$$\hat{H}_{harm}^\theta = -\frac{1}{2(2\pi)^2} \sum_{i,j=1}^{b_1(X)} (h_X^{(1)})_{ij}^{-1} \nabla_i \nabla_j + \frac{(2\pi)^2}{2} \sum_{k,l=1}^{b_2(X)} (h_X^{(2)})_{kl} m_k m_l, \quad (4.12)$$

where $\nabla_i := \frac{\partial}{\partial q_i} - 2\pi\sqrt{-1}\theta_i$ is the covariant derivative. This system can be interpreted as quantum theory of a classical particle in $b_1(X)$ dimensions coupled to a constant (functional) electric field $\vec{\theta}$ and moving in an external constant potential determined by the topological sector $\vec{m} \in \mathbb{Z}^{b_2(X)}$.

The energy eigenstates of \hat{H}_{harm}^θ are the wave functions $\psi_l(q_1, \dots, q_{b_1(X)}) = e^{2\pi\sqrt{-1} \sum_{i=1}^{b_1(X)} l_i q_i}$ with eigenvalues

$$\varepsilon_{\vec{l}, \vec{m}}^\theta = \frac{1}{2} \sum_{i,j=1}^{b_1(X)} (h_X^{(1)})_{ij}^{-1} (l_i - \theta_i)(l_j - \theta_j) + \frac{(2\pi)^2}{2} \sum_{\substack{k,l=1 \\ k < l}}^{b_2(X)} (h_X^{(2)})_{kl} m_k m_l. \quad (4.13)$$

In the previous section we introduced the topological action S_θ and argued that this term accounts for the inequivalent quantum theories caused by the topologically non-trivial configuration space. Now we will justify this argument: Let us introduce the unitary operator U_θ by

$$\tilde{\psi}(\vec{q}) := (U_\theta \psi)(\vec{q}) = e^{-2\pi\sqrt{-1} \sum_{j=1}^{b_1(X)} \theta_j q_j} \psi(\vec{q}), \quad \vec{q} = (q_1, \dots, q_{b_1(X)}). \quad (4.14)$$

Since $U_\theta \nabla_i U_\theta^{-1} = \frac{\partial}{\partial q_i}$, the θ -dependent term in (4.12) can be removed giving rise to a new Hamilton operator $\hat{H}_{harm} := U_\theta \hat{H}_{harm}^\theta U_\theta^{-1}$ with eigenstates $\tilde{\psi}_{\vec{l}}(\vec{q}) := U_\theta \psi_{\vec{l}}$. Since \hat{H}_{harm}^θ and \hat{H}_{harm} have the same spectrum, the corresponding quantum theories are equivalent. However, the wave functions $\tilde{\psi}$ are no longer single-valued, since $\tilde{\psi}(\vec{q} + \vec{\alpha}) = e^{-2\pi\sqrt{-1} \sum_{j=1}^{b_1(X)} \alpha_j \theta_j} \tilde{\psi}(\vec{q})$ for $\vec{\alpha} \in \mathbb{Z}^{b_1(X)}$. For each fixed $\vec{\theta} \in \mathbb{R}^{b_1(X)}$ these wave functions can be regarded as sections of the line-bundle $\mathcal{L}^\theta := \mathbb{R}^{b_1(X)} \times_{\vec{\theta}} \mathbb{C}$ over $\mathbb{T}^{b_1(X)}$, which is associated to the universal covering $\mathbb{R}^{b_1(X)} \rightarrow \mathbb{T}^{b_1(X)}$ via the unitary irreducible representation $\vec{\alpha} \mapsto e^{2\pi\sqrt{-1} \vec{\alpha} \vec{\theta}}$. Thus different choices for $\vec{\theta} \notin \mathbb{Z}^{b_1(X)}$ lead to inequivalent quantum theories.

Now we are going to quantize the transversal modes: Let $\{\varpi_{\alpha_1} | \alpha_1 \in J_1\}$ denote an orthonormal basis of eigenforms satisfying $\Delta_1^X |_{imd_2^*} \varpi_{\alpha_1} = \nu_{\alpha_1} (\Delta_1^X |_{imd_2^*}) \varpi_{\alpha_1}$, where each eigenvalue $\nu_{\alpha_1} (\Delta_1^X |_{imd_2^*})$ appears as often as its multiplicity. With respect to the decompositions $\tau = \sum_{\alpha_1 \in J_1} \tau_{\alpha_1} \varpi_{\alpha_1}$ and $\Upsilon = \sum_{\alpha_1 \in J_1} \Upsilon_{\alpha_1} \varpi_{\alpha_1}$ one obtains

$$H_{trans} = \sum_{\alpha_1 \in J_1} \left[\frac{1}{2} |\Upsilon_{\alpha_1}|^2 + \frac{1}{2} \nu_{\alpha_1} (\Delta_1^X |_{imd_2^*}) |\tau_{\alpha_1}|^2 \right], \quad (4.15)$$

which is nothing but the (well-known) Hamiltonian of an infinite number of harmonic oscillators with frequencies $\sqrt{\nu_{\alpha_1} (\Delta_1^X |_{imd_2^*})}$. The coefficients are $\tau_{\alpha_1} = \langle A - A_0, \varpi_{\alpha_1} \rangle$, $\Upsilon_{\alpha_1} = \langle \Upsilon, \varpi_{\alpha_1} \rangle$, having the properties $\bar{\tau}_{\alpha_1} = -\tau_{\alpha_1}$ and $\bar{\Upsilon}_{\alpha_1} = -\Upsilon_{\alpha_1}$. The non-vanishing Poisson brackets are $\{\tau_{\alpha_1}, \bar{\Upsilon}_{\alpha'_1}\} = -\langle \varpi_{\alpha_1}, \varpi_{\alpha'_1} \rangle = \delta_{\alpha_1, \alpha'_1}$, which in the quantum theory are replaced by the commutators $[\tau_{\alpha_1}, \bar{\Upsilon}_{\alpha'_1}] = \sqrt{-1} \delta_{\alpha_1, \alpha'_1}$. The spectrum of (4.15) is easily obtained in the Fock representation by introducing annihilation and creation operators

$$\begin{aligned} b_{\alpha_1} &:= (4\nu_{\alpha_1} (\Delta_1^X |_{imd_2^*}))^{-\frac{1}{4}} \left[\sqrt{-1} \sqrt{\nu_{\alpha_1} (\Delta_1^X |_{imd_2^*})} \tau_{\alpha_1} - \Upsilon_{\alpha_1} \right] \\ b_{\alpha_1}^\dagger &:= (4\nu_{\alpha_1} (\Delta_1^X |_{imd_2^*}))^{-\frac{1}{4}} \left[-\sqrt{-1} \sqrt{\nu_{\alpha_1} (\Delta_1^X |_{imd_2^*})} \bar{\tau}_{\alpha_1} - \bar{\Upsilon}_{\alpha_1} \right], \end{aligned} \quad (4.16)$$

which have the non-vanishing commutators $[b_{\alpha_1}, b_{\alpha'_1}^\dagger] = \delta_{\alpha_1, \alpha'_1}$. The transverse Hamilton operator admits then the following familiar form

$$\hat{H}_{trans} = \sum_{\alpha_1 \in J_1} \sqrt{\nu_{\alpha_1} (\Delta_1^X |_{imd_2^*})} \left[b_{\alpha_1}^\dagger b_{\alpha_1} + \frac{1}{2} \right], \quad (4.17)$$

which admits the energy eigenvalues $\varepsilon_{k_{\alpha_1}}^{trans} = \sqrt{\nu_{\alpha_1}(\Delta_1^X|_{imd_2^*})} (k_{\alpha_1} + \frac{1}{2})$, where $k_{\alpha_1} \in \mathbb{N}_0$.

In order to determine the thermal partition function, we have to take the different topological sectors into account. These are labelled by the first Chern class (see (3.19)), namely

$$c_1(\underline{Q}) = i_X^* c_1(Q) = \sum_{j=1}^{b_2(X)} m_j \eta_j^{(2)} + \sum_{k=1}^w y_k t_k^{(2)} \in H^2(X; \mathbb{Z}). \quad (4.18)$$

Since \hat{H}_{harm}^θ depends on the free part of $H^2(X; \mathbb{Z})$ only, one obtains

$$\begin{aligned} \hat{\mathcal{Z}}^\theta(\beta; V) &= \sum_{c_1(\underline{Q}) \in H^2(X; \mathbb{Z})} Tr \left(e^{-\beta \hat{H}^\theta} \right) \\ &= \sum_{\vec{m} \in \mathbb{Z}^{b_2(X)}} \sum_{\vec{l} \in \mathbb{Z}^{b_1(X)}} \prod_{\alpha_1 \in J_1} \sum_{k_{\alpha_1} \in \mathbb{N}_0} e^{-\beta \varepsilon_{\vec{l}, \vec{m}}^\theta} e^{-\beta \varepsilon_{k_{\alpha_1}}^{trans}} |Tor H^2(X; \mathbb{Z})|. \end{aligned} \quad (4.19)$$

where the trace has been calculated by summing over all physical eigenstates. The free energy is then given by

$$\begin{aligned} \hat{\mathcal{F}}^\theta(\beta; V) &= \frac{1}{2} \sum_{\alpha_1 \in J_1} \sqrt{\nu_{\alpha_1}(\Delta_1^X|_{imd_2^*})} + \frac{1}{\beta} \sum_{\alpha_1 \in J_1} \ln \left[1 - e^{-\beta \sqrt{\nu_{\alpha_1}(\Delta_1^X|_{imd_2^*})}} \right] \\ &\quad - \frac{1}{\beta} \ln \Theta_{b_1(X)} \left[\begin{matrix} \vec{\theta} \\ 0 \end{matrix} \right] \left(0 \middle| \frac{\sqrt{-1}}{2\pi} \beta (h_X^{(1)})^{-1} \right) - \frac{1}{\beta} \ln \Theta_{b_2(X)} \left(0 \middle| 2\pi \sqrt{-1} \beta h_X^{(2)} \right) \\ &\quad - \frac{1}{\beta} \ln |Tor H^2(X; \mathbb{Z})|. \end{aligned} \quad (4.20)$$

Obviously, the first term in (4.20), which represents the vacuum energy ε_{vac} of the transverse modes of the electromagnetic field is infinite and requires a regularization. We choose zeta-function regularization and introduce the following function in $s \in \mathbb{C}$

$$\tilde{\varepsilon}(s) := \frac{1}{2} \sum_{\alpha_1} \frac{(\nu_{\alpha_1}(\Delta_1^X|_{imd_2^*}))^{\frac{1}{2}}}{(\tilde{\mu}^{-2} \nu_{\alpha_1}(\Delta_1^X|_{imd_2^*}))^{s+\frac{1}{2}}} = \frac{1}{2} \tilde{\mu}^{2s+1} \zeta(s; \Delta_1^X|_{imd_2^*}), \quad (4.21)$$

where $\tilde{\mu}$ is a scale parameter with mass dimension $[\tilde{\mu}] = 1$. This step is necessary to assign $\tilde{\varepsilon}$ the correct mass dimension. A natural choice for the vacuum energy could be to take $\varepsilon_{vac} = \tilde{\varepsilon}(-\frac{1}{2})$. However, this would be reasonable unless $\zeta(s; \Delta_1^X|_{imd_2^*})$ becomes divergent in $s = -\frac{1}{2}$. In order to account for this case as well, we follow [35] and define the regularized vacuum energy of the transverse modes as finite part of $\tilde{\varepsilon}$, i.e.

$$\varepsilon_{vac}^{reg} := FP_{s=-\frac{1}{2}}[\tilde{\varepsilon}(s)]. \quad (4.22)$$

This regularization scheme amounts to remove the pole from (4.21). Since $FP_{s=s_0}[f_1(s)f_2(s)] = f_1(s_0)FP_{s=s_0}[f_2(s)] + f_1'(s_0)Res_{s=s_0}[f_2(s)]$ holds for a function f_1 being holomorphic in s_0 and f_2 being meromorphic with simple pole at the same point s_0 , one finally gets

$$\varepsilon_{vac}^{reg} = \frac{1}{2} FP_{s=-\frac{1}{2}}[\zeta(s; \Delta_1^X|_{imd_2^*})] + \frac{1}{2} Res_{s=-\frac{1}{2}}[\zeta(s; \Delta_1^X|_{imd_2^*})] \ln \tilde{\mu}^2, \quad (4.23)$$

which represents the finite vacuum free energy of the transverse modes of the Maxwell field. Let us now substitute the first term in (4.20) by ε_{vac}^{reg} and choose $\tilde{\mu} = \frac{e\mu}{2}$, then one finds

$$\hat{\mathcal{F}}^\theta(\beta; V) = \mathcal{F}^\theta(\beta; V). \quad (4.24)$$

Hence we have explicitly verified that both quantization schemes are equivalent. Let us stress the fact that keeping the field independent Jacobian, taking the quotient by the volume of the total gauge group and finally summing over the different topological sectors have been the main steps in the functional integral scheme to obtain this equality with the Hamiltonian approach. In the remainder of this paper the caret on the free energy will thus be omitted.

The normalization scale μ expresses the ambiguity of the free energy which occurs whenever $Res_{s=-\frac{1}{2}} [\zeta(s; \Delta_1^X|_{imd_2^*})] \neq 0$. In that case renormalization issues have to be considered. Alternatively, this could be stated as follows: If $\zeta(s; \Delta_1^X|_{imd_2^*})$ is finite for $s = -\frac{1}{2}$, the free energy is uniquely determined and the scale dependency must disappear.

In the case of massless scalar fields at finite temperature the relation between these two quantization schemes and in particular the question regarding the zero modes of the kinetic operator have been discussed some time ago in Refs. [18, 19, 20] and recently in Ref. [23].

In order to split $\mathcal{F}^\theta(\beta; V)$ into a temperature independent term $\mathcal{F}_0^\theta(V)$ and a temperature dependent contribution $\mathcal{F}_{temp}^\theta(\beta; V)$, we use (A.1) to rewrite

$$\begin{aligned} \Theta_{b_1(X)} \begin{bmatrix} \vec{\theta} \\ 0 \end{bmatrix} \left(0 \middle| \frac{\sqrt{-1}}{2\pi} \beta (h_X^{(1)})^{-1} \right) &= \\ &= e^{-\frac{1}{2} \beta \langle \vec{\theta} \rangle^\dagger (h_X^{(1)})^{-1} \langle \vec{\theta} \rangle} \Theta_{b_1(X)} \left(\frac{1}{2\pi\sqrt{-1}} \beta (h_X^{(1)})^{-1} \langle \vec{\theta} \rangle \middle| \frac{\sqrt{-1}}{2\pi} \beta (h_X^{(1)})^{-1} \right). \end{aligned} \quad (4.25)$$

According to the modular property of the Riemann Theta function (A.2), $\vec{\theta}$ can be replaced by the translated vector $\langle \vec{\theta} \rangle$ whose j -th component is defined by $\langle \theta_j \rangle := \text{sgn}(\theta_j) \min_{m_j \in \mathbb{Z}} |m_j - \theta_j|$, where $j = 1, \dots, b_1(X)$. Hence the components of $\langle \vec{\theta} \rangle$ are restricted to $|\langle \theta_j \rangle| \leq \frac{1}{2}$ for all j .

The expression for the free energy of the quantum Maxwell field at finite temperature admits now its final form

$$\begin{aligned} \mathcal{F}^\theta(\beta; V) &= \mathcal{F}_0^\theta(V) + \mathcal{F}_{temp}^\theta(\beta; V) \\ &= \frac{1}{2} FP_{s=-\frac{1}{2}} [\zeta(s; \Delta_1^X|_{imd_2^*})] + \frac{1}{2} Res_{s=-\frac{1}{2}} [\zeta(s; \Delta_1^X|_{imd_2^*}) \ln \left(\frac{e\mu}{2} \right)^2] \\ &\quad + \frac{1}{2} \sum_{j,k=1}^{b_1(X)} (h_X^{(1)})_{jk}^{-1} \langle \theta_j \rangle \langle \theta_k \rangle + \frac{1}{\beta} \sum_{\alpha_1 \in J_1} \ln \left[1 - e^{-\beta \sqrt{\nu_{\alpha_1}(\Delta_1^X|_{imd_2^*})}} \right] \\ &\quad - \frac{1}{\beta} \ln \Theta_{b_1(X)} \left(\frac{1}{2\pi\sqrt{-1}} \beta (h_X^{(1)})^{-1} \langle \vec{\theta} \rangle \middle| \frac{\sqrt{-1}}{2\pi} \beta (h_X^{(1)})^{-1} \right) \\ &\quad - \frac{1}{\beta} \ln \Theta_{b_2(X)} \left(0 \middle| 2\pi\sqrt{-1} \beta h_X^{(2)} \right) - \frac{1}{\beta} \ln |Tor H^2(X; \mathbb{Z})|. \end{aligned} \quad (4.26)$$

This formula for the regularized (total) free energy of the photon gas confined to a closed manifold X is the main result of the present paper and - to the best of our knowledge - has not

been stated before. It exhibits clearly how the topology of X affects both the vacuum energy and the thermodynamic structure of the system. The vacuum energy \mathcal{F}_0^θ is the sum of the regularized vacuum energy of the transverse modes and the ground-state energy of \hat{H}_{harm}^θ (see (4.12)). The latter vanishes whenever $\vec{\theta} \in \mathbb{Z}^{b_1(X)}$. In any case the Laplace operators appearing in (4.26) are correspondingly restricted in order to rule out any zero-modes. This is the consequence of the construction of the partition function by using either a family of local trivializations in the functional integral approach or an appropriate parametrization of the true phase space in the Hamiltonian scheme.

The free energy is unique only if the zeta function converges at $s = -\frac{1}{2}$. On the other hand, if the first and second cohomology group of X vanish, the Riemann Theta functions as well as the temperature independent θ -vacuum term disappear and the free energy is completely determined by the transverse modes. Under these conditions the free energy of the quantum Maxwell theory would be indeed a multiple of the free energy of a massless scalar gas. The n -sphere $X = \mathbb{S}^n$ with $n \neq 1, 2$ is a typical example for such a configuration. In so far we gave a proof for the statement argued in [30, 31].

In the case $n = 1$, the transverse modes are absent so that the free energy is exclusively governed by the harmonic component. For $X = \mathbb{S}^2$ the θ -states are absent, but since $H^2(\mathbb{S}^2; \mathbb{Z}) \cong \mathbb{Z}$, the topological sectors contribute additionally to the thermal excitations.

Let us now compare our result with the free energy thermal contributions of a photon gas in flat Euclidean space confined to a very large box in Euclidean space. In fact, when considering the thermal excitations in the infinite volume limit, the index α_1 in the fourth term in the second equation in (4.26) becomes the continuous n -dimensional wave vector \vec{k} . Hence all finite size and topological effects are neglected. Furthermore, the sum is replaced by an integration with respect to the measure $\frac{d^n \vec{k}}{(2\pi)^n}$. Using [43] one gets finally for the thermal part of the free energy density in that limit

$$(n-1) \int_{\mathbb{R}^n} \frac{d^n \vec{k}}{(2\pi)^n} \ln \left(1 - e^{-\beta |\vec{k}|} \right) = -(n-1) \pi^{-\frac{n+1}{2}} \Gamma\left(\frac{n+1}{2}\right) \zeta_R(n+1) \beta^{-(n+1)}, \quad (4.27)$$

where the factor $n-1$ is the number of independent degrees of freedom. Since the topology of Euclidean space is trivial the Riemann Theta functions are absent. Eq. (4.27) is the well-known Stefan-Boltzmann term of black-body radiation.

Let us emphasize that in our study the photon gas is confined to a closed spatial manifold X . Evidently, there is no space "outside" of X , which could contribute neither to the vacuum energy nor to the thermodynamic excitations of the system. In that sense, (4.26) represents the intrinsic free energy of the photon gas. This has to be distinguished from configurations where a partition separates an inside and an outside region in an ambient space. The resulting (Casimir) force exerted by the photon gas on that partition is then caused by the difference - the so-called *Casimir free energy* - between the free energy in the inside and outside region. Typical examples are infinite parallel plates [10], rectangular cavities [23, 24] and piston geometries [45, 46, 47].

As was stated above, in our case the vacuum (Casimir) energy depends, in general, on the normalization scale μ . However, if the electromagnetic field is confined to a compact and connected cavity with smooth perfectly conducting boundary within \mathbb{R}^3 , the corresponding Casimir energy was shown to be finite [48]. This is based on an explicit computation of the heat kernel expansion of the corresponding Laplace operators and the fact that relevant contributions from

the inside and the outside of the cavity cancel. Hence for that configuration no renormalization is necessary.

In the remainder of this section we want to determine the high-temperature limit of the free energy. The starting expression will be (3.35) together with (3.38). Let us define the auxiliary quantity

$$K(s; D') := 2 \sum_{k=1}^{\infty} \sum_{\alpha \in J'} \left[\left(\frac{2\pi k}{\beta} \right)^2 + \nu_{\alpha}(D') \right]^{-s}, \quad (4.28)$$

where the second sum runs over all eigenvalues of D' , thus excluding the zero modes of D . By separating the zero-modes, the quantity $\mathcal{I}(s; D)$ in (3.36) can be alternatively rewritten in the form

$$\mathcal{I}(s; D) = \zeta(s; D') + 2(\dim \ker D) \left(\frac{\beta}{2\pi} \right)^{2s} \zeta_R(2s) + K(s; D'). \quad (4.29)$$

The Mellin transformation of $K(s; D')$ reads

$$K(s; D') = \frac{2}{\Gamma(s)} \int_0^{\infty} dt \, t^{s-1} \sum_{k=1}^{\infty} e^{-(\frac{2\pi}{\beta})^2 k^2 t} \sum_{\alpha \in J} e^{-\nu_{\alpha}(D')t} \quad (4.30)$$

for $\Re(s) > \frac{n}{2}$ but can be analytically continued elsewhere [49]. Its asymptotic expansion for $\beta \rightarrow 0$ can be obtained by substituting the heat kernel expansion (3.28) for the restricted operator D' with corresponding coefficients, denoted by $a_m(D')$. The integration term by term gives

$$K(s; D') \simeq 2 \left(\frac{2\pi}{\beta} \right)^{-2s} \sum_{m=0}^{\infty} a_m(D') \left(\frac{2\pi}{\beta} \right)^{n-m} \frac{\Gamma(s + \frac{m-n}{2}) \zeta_R(2s + m - n)}{\Gamma(s)}. \quad (4.31)$$

In order to perform the limit $s \rightarrow 0$ of $K(s; D')$ and of its derivative respectively, we notice that the function $\Gamma(s + \frac{m-n}{2}) \zeta_R(2s + m - n)$ has simple poles at $m = n$ and $m = n + 1$. Using that $E_1(s; 1) = 2\zeta_R(2s)$, where E_1 is the Epstein zeta function (B.1) in one dimension, the reflection formula (B.5) provides the analytic continuation of that function. If we take the Laurent expansion of $\zeta_R(s)$ at the pole $s = -1$ and the series expansion for $\frac{1}{\Gamma(s)}$ (see above), a lengthy calculation yields

$$\begin{aligned} \frac{1}{2} \frac{d}{ds} \Big|_{s=0} K(s; D') &= \sum_{\substack{m=0 \\ m \neq n \\ m \neq n+1}}^{\infty} a_m(D') 2^{n-m} \pi^{-\frac{1}{2}} \beta^{m-n} \Gamma\left(\frac{n+1-m}{2}\right) \zeta_R(n+1-m) \\ &\quad + a_{n+1}(D') \frac{\beta}{2\pi^{\frac{1}{2}}} \left(\ln \left(\frac{\beta}{4\pi} \right) + \gamma \right) - a_n(D') \ln \beta, \end{aligned} \quad (4.32)$$

where (once again) γ is the Euler constant. From $a_m(D) = a_m(D') + \delta_{m,n}(\dim \ker D)$ and the spectrum of Δ_p^X ($p = 0, 1$), it follows that

$$a_m(\Delta_1^X|_{imd_2^*}) = a_m(\Delta_1^X) - a_k(\Delta_0^X) + (1 - b_1(X))\delta_{m,n}. \quad (4.33)$$

Using (4.31), (4.32), (4.33), the explicit form $a_0(\Delta_p^X) = (4\pi)^{-\frac{n}{2}} \frac{n!}{p!(n-p)!} V$ [40] and finally applying the duality formula (A.3) to the Riemann Theta function $\Theta_{b_2(X)}(0|\dots)$, one obtains from (3.35) the high-temperature asymptotic expansion for the free energy in terms of the coefficients $a_m(\Delta_1^X|_{imd_2^*})$, namely

$$\begin{aligned}
\mathcal{F}^\theta(\beta; V) \simeq & - (n-1) \pi^{-\frac{n+1}{2}} \Gamma\left(\frac{n+1}{2}\right) \zeta_R(n+1) \beta^{-(n+1)} V \\
& - \sum_{\substack{m=1 \\ m \neq n \\ m \neq n+1}}^{\infty} a_m(\Delta_1^X|_{imd_2^*}) 2^{n-m} \pi^{-\frac{1}{2}} \beta^{m-n-1} \Gamma\left(\frac{n+1-m}{2}\right) \zeta_R(n+1-m) \\
& - \frac{1}{2\beta} \left[\zeta'(0; \Delta_1^X|_{imd_2^*}) + \ln \left(\frac{\det(2\pi h_X^{(1)})}{\det(2\pi h_X^{(2)})} |Tor H^2(X; \mathbb{Z})|^2 \right) \right] \\
& + \frac{1}{2\beta} [2a_n(\Delta_1^X|_{imd_2^*}) + b_1(X) + b_2(X)] \ln \beta \\
& + Res_{s=-\frac{1}{2}} [\zeta(s; \Delta_1^X|_{imd_2^*})] \left(\ln \left(\frac{\mu\beta}{4\pi} \right) + \gamma \right) \\
& - \frac{1}{\beta} \ln \Theta_{b_1(X)} \left(\langle \vec{\theta} \rangle | 2\pi \sqrt{-1} \beta^{-1} h_X^{(1)} \right) - \frac{1}{\beta} \ln \Theta_{b_2(X)} \left(0 | \frac{\sqrt{-1}}{2\pi} \beta^{-1} (h_X^{(2)})^{-1} \right).
\end{aligned} \tag{4.34}$$

The first term highlights the familiar Stefan Boltzmann term in n spatial dimensions (see (4.27)) and the remaining terms represent the modifications caused by the topology of X . If $\zeta(s; \Delta_1^X|_{imd_2^*})$ is regular at $s = -\frac{1}{2}$, the μ -dependent term vanishes as expected and yields an unambiguous result. A corresponding formula for a gas of massless scalar fields on a compact manifold with and without boundary has been firstly derived in [11] and generalized in [12]. The high temperature limit for massless scalar fields in different topologies have been discussed by many authors [17, 18, 19, 20, 23] since then.

When concerning the Casimir contribution to the free energy (i.e. the Casimir free energy) in the case of material boundaries (e.g. parallel plates, pistons, rectangular box) all terms, which are of quantum origin like the Stefan-Boltzmann term are cancelled giving rise to the classical limit at high temperature [24, 46, 50].

Quite recently, the thermal Casimir effect was reconsidered for several types of fields in the static Einstein and closed Friedmann universe [51, 52]. In order to obtain the Casimir free energy, it was proposed to use the renormalization scheme, which is usually applied in the case of material boundaries, also for the treatment of the topological Casimir effect at finite temperature. In fact, the Casimir free energy is the difference between the free energy of the topologically non-trivial manifold and the free energy of the tangential Euclidean/Minkowski space both filled with thermal radiation. Thereby not only the zero-temperature vacuum energy but even the finite-temperature contributions are renormalized. As a result the Casimir free energy tends to the classical limit at high temperatures, where the leading term is linear in temperature.

5 The equation of state

Once the free energy \mathcal{F}^θ is determined, the main thermodynamic functions can be computed by the following formulae:

$$U^\theta(\beta, V) = \frac{\partial}{\partial \beta} \left(\beta \mathcal{F}^\theta(\beta, V) \right), \quad S^\theta(\beta, V) = \beta^2 \frac{\partial}{\partial \beta} \mathcal{F}^\theta(\beta, V), \quad P^\theta(\beta, V) = -\frac{\partial}{\partial V} \mathcal{F}^\theta(\beta, V). \quad (5.1)$$

Here U^θ is the internal energy, S^θ denotes the entropy and P^θ is the pressure of the photon gas. Since the thermodynamic functions are periodic under translations $\vec{\theta} \mapsto \vec{\theta} + \vec{m}$, for any $\vec{m} \in \mathbb{Z}^{b_1(X)}$, we can replace $\vec{\theta}$ by $\langle \vec{\theta} \rangle$.

Now we want to study the behavior of the free energy under a constant scale transformation $\beta \mapsto \lambda \beta$ and $V \mapsto \lambda^n V$ with $\lambda \in \mathbb{R}$. Equivalently, this can be regarded as scale transformation of the metric $g \mapsto \lambda^2 g$. A direct calculation yields

$$\begin{aligned} vol_X &\mapsto \lambda^n vol_X, & \nu_k^{(i)}(\Delta_r^{\mathbb{T}_\beta^1}) &\mapsto \lambda^{-2} \nu_k^{(i)}(\Delta_r^{\mathbb{T}_\beta^1}) \\ h_X^{(1)} &\mapsto \lambda^{n-2} h_X^{(1)}, & \nu_{l_j}^{(j)}(\Delta_s^X) &\mapsto \lambda^{-2} \nu_{l_j}^{(j)}(\Delta_s^X) \\ h_X^{(2)} &\mapsto \lambda^{n-4} h_X^{(2)}, \end{aligned} \quad (5.2)$$

with $r, s = 0, 1$. As a consequence, the free energy displays the following transformation behavior

$$\begin{aligned} \mathcal{F}^\theta(\lambda \beta, \lambda^n V) &= \frac{1}{\lambda} \mathcal{F}^\theta(\beta, V) + \frac{\ln \lambda}{\lambda} Res_{s=-\frac{1}{2}} [\zeta(s; \Delta_1^X |_{imd_2^*})] \\ &+ \frac{1}{\lambda \beta} \ln \left(\frac{\Theta_{b_1(X)} \left[\begin{smallmatrix} \langle \vec{\theta} \rangle \\ 0 \end{smallmatrix} \right] \left(0 | \frac{\sqrt{-1}}{2\pi} \beta (h_X^{(1)})^{-1} \right) \Theta_{b_2(X)} \left(0 | 2\pi \sqrt{-1} \beta h_X^{(2)} \right)}{\Theta_{b_1(X)} \left[\begin{smallmatrix} \langle \vec{\theta} \rangle \\ 0 \end{smallmatrix} \right] \left(0 | \frac{\sqrt{-1}}{2\pi} \beta (h_X^{(1)})^{-1} \lambda^{3-n} \right) \Theta_{b_2(X)} \left(0 | 2\pi \sqrt{-1} \beta h_X^{(2)} \lambda^{n-3} \right)} \right), \end{aligned} \quad (5.3)$$

showing that the free energy does no longer transform homogenously of degree -1 . In fact, this shows that the free energy is not an extensive quantity. The violation of the scale invariance is caused on the one hand by the introduction of a scale ambiguity in the regularization of the vacuum energy and on the other hand by the harmonic modes of the Maxwell field. This anomaly has an implication for the equation of state. In fact, taking the derivative of (5.3) by λ , setting $\lambda = 1$ and using (5.1) yields

$$\mathcal{F}^\theta(\beta, V) = n P^\theta(\beta, V) V - \beta^{-1} S^\theta(\beta, V) + \Gamma^\theta(\beta, V), \quad (5.4)$$

with

$$\begin{aligned} \Gamma^\theta(\beta; V) &= Res_{s=-\frac{1}{2}} [\zeta(s; \Delta_1^X |_{imd_2^*})] - \frac{n-3}{2} \sum_{j,k=1}^{b_1(X)} (h_X^{(1)})_{jk}^{-1} \langle \theta_j \rangle \langle \theta_k \rangle \\ &+ \frac{n-3}{\beta} \frac{\partial}{\partial \lambda} \Big|_{\lambda=1} \ln \left(\frac{\Theta_{b_1(X)} \left(\frac{\lambda}{2\pi \sqrt{-1}} \beta (h_X^{(1)})^{-1} \langle \vec{\theta} \rangle | \lambda \frac{\sqrt{-1}}{2\pi} \beta (h_X^{(1)})^{-1} \right)}{\Theta_{b_2(X)} \left(0 | 2\lambda \pi \sqrt{-1} \beta h_X^{(2)} \right)} \right). \end{aligned} \quad (5.5)$$

Together with the general relation $\mathcal{F}^\theta(\beta, V) = U^\theta(\beta, V) - \beta^{-1}S^\theta(\beta, V)$ one gets the following modified equation of state

$$P^\theta(\beta, V) = \frac{1}{nV} \left[U^\theta(\beta, V) - \Gamma^\theta(\beta, V) \right]. \quad (5.6)$$

Thus we have shown that the equation of state of a photon gas on a topologically non-trivial and compact spatial manifold differs from the conventional one in the large volume limit by the anomalous term Γ^θ . Irrespective of the topology of X , the "topological" terms in Γ^θ vanishes for $n = 3$. If in addition the zeta function is finite for $s = -\frac{1}{2}$, then we obtain the familiar relation $P^\theta V = \frac{1}{3}U^\theta$ in three dimensions.

6 Explicit results for the n -torus \mathbb{T}^n

In the remainder of this paper we want to consider the photon gas confined to a n -dimensional torus $X = \mathbb{T}^n$ in more detail. The n -torus is equipped with a flat metric $\gamma = \sum_{i=1}^n L_i^2 dt^i \otimes dt^i$, where the sequence $(t^i)_{i=1}^n$ denotes the local coordinates of \mathbb{T}^n and L_i is the length in the i -th direction, so that the spatial volume is $V = \prod_{i=1}^n L_i$. This configuration can be equivalently realized as field theory in a n -dimensional rectangular box $X = [0, L_1] \times \dots \times [0, L_n]$ subjected to periodic spatial boundary conditions.

Massless scalar fields in a box with periodic boundary conditions at finite temperature were studied in [13, 17, 22, 23] based on Epstein zeta function regularization and in [22] using a multidimensional cut-off. An analysis of the finite-size effects in a universe with toroidal topology was presented in [15].

We would like to stress once again that in the case of a non-vanishing boundary (where the fields in the rectangular box are satisfying Dirichlet or von Neumann boundary conditions) one has to take into account the contribution both from the inside of the cavity as well as from the outside region [24] for the calculation of the Casimir free energy.

A Betti basis for $\mathcal{H}_{\mathbb{Z}}^1(\mathbb{T}^n)$ is given by $\rho_j^{(1)} = \frac{1}{L_j} \hat{i}_j^* \text{vol}_{\mathbb{T}^n}^\gamma = dt^j$, where $\hat{i}_j: \mathbb{T}^1 \hookrightarrow \mathbb{T}^n$ is the canonical inclusion of the j -th position and $j = 1, \dots, n$. Correspondingly, the dual Betti basis for $\mathcal{H}_{\mathbb{Z}}^{n-1}(\mathbb{T}^n)$ is provided by $\rho_j^{(n-1)} = (-1)^{j-1} \rho_1^{(1)} \wedge \dots \wedge \widehat{\rho_j^{(1)}} \wedge \dots \wedge \rho_n^{(1)}$, where the caret denotes omission of the respective element. By construction $\int_{\mathbb{T}^n} \rho_j^{(1)} \wedge \rho_k^{(n-1)} = \delta_{jk}$. Furthermore $\rho_{jk}^{(2)} = \frac{1}{L_j L_k} \hat{i}_j^* \text{vol}_{\mathbb{T}^n}^\gamma \wedge \hat{i}_k^* \text{vol}_{\mathbb{T}^n}^\gamma = dt^j \wedge dt^k$ induces a Betti basis for $\mathcal{H}_{\mathbb{Z}}^2(\mathbb{T}^n)$, where $j, k = 1, \dots, n$ and $j < k$. As a consequence, $\mathcal{H}^p(\mathbb{T}^n)$ ($p = 1, 2$) admit the following metrics

$$\begin{aligned} h_{\mathbb{T}^n}^{(1)} &= \text{diag} \left(\frac{V}{L_1^2}, \dots, \frac{V}{L_n^2} \right) \\ h_{\mathbb{T}^n}^{(2)} &= \text{diag} \left(\frac{V}{L_1^2 L_2^2}, \frac{V}{L_1^2 L_3^2}, \dots, \frac{V}{L_{n-1}^2 L_n^2} \right). \end{aligned} \quad (6.1)$$

In order to obtain the free energy we have to calculate the spectrum and the zeta-function of $\Delta_1^{\mathbb{T}^n}|_{\text{imd}_2^*}$. It is easy to prove that the eigenvalues of $\Delta_1^{\mathbb{T}^n}|_{\text{imd}_2^*}$ are the sequence of real numbers $\nu_{\vec{m}}(\Delta_1^{\mathbb{T}^n}|_{\text{imd}_2^*}) = \sum_{j=1}^n \left(\frac{2\pi m_j}{L_j} \right)^2$, parametrized by $\vec{m} \in \mathbb{Z}_0^n := \mathbb{Z}^n \setminus \{0\}$. For each $\vec{m} \in \mathbb{Z}_0^n$, we introduce vielbeins $\epsilon_{r,j,\vec{m}}$, which are labelled by r and have components indexed by j , satisfying

$$\sum_{j=1}^n \epsilon_{rj,\vec{m}} \epsilon_{r'j,\vec{m}} = \delta_{rr'}, \quad \epsilon_{nj,\vec{m}} = \left(\nu_{\vec{m}}(\Delta_1^{\mathbb{T}^n}|_{imd_2^*}) \right)^{-\frac{1}{2}} \frac{2\pi m_j}{L_j}, \quad j, r = 1, \dots, n. \quad (6.2)$$

Let us define the inner product $(\varphi, \chi)_\gamma := \langle \varphi, \bar{\chi} \rangle_\gamma$ for $\varphi, \chi \in \Omega^p(\mathbb{T}^n, \mathbb{C})$, where $\bar{\chi}$ is the complex conjugate of χ , then the following 1-forms

$$\varpi_{r,\vec{m}} := \sum_{j=1}^n \epsilon_{rj,\vec{m}} \left(\frac{L_j^2}{V} \right)^{\frac{1}{2}} e^{2\pi\sqrt{-1}\sum_{i=1}^n m_i t_i} dt^j, \quad \vec{m} \in \mathbb{Z}_0^n, \quad r = 1, \dots, n \quad (6.3)$$

satisfy $(\varpi_{r,\vec{m}}, \varpi_{r',\vec{m}'})_\gamma = \delta_{rr'} \delta_{\vec{m},\vec{m}'}$. A direct calculation gives

$$d_2^* G_2 d_1 \varpi_{r,\vec{m}} = \begin{cases} \varpi_{r,\vec{m}} & \text{if } r = 1, \dots, n-1 \\ 0 & \text{if } r = n, \end{cases} \quad (6.4)$$

and

$$\Delta_1^{\mathbb{T}^n}|_{imd_2^*} \varpi_{r,\vec{m}} = \nu_{\vec{m}}(\Delta_1^{\mathbb{T}^n}|_{imd_2^*}) \varpi_{r,\vec{m}}, \quad \vec{m} \in \mathbb{Z}_0^n, \quad r = 1, \dots, n-1, \quad (6.5)$$

proving that the set $\{\varpi_{r,\vec{m}} | r = 1, \dots, n-1, \vec{m} \in \mathbb{Z}_0^n\}$ provides an orthonormal basis of eigenforms of $\Delta_1^{\mathbb{T}^n}|_{imd_2^*}$ with respect to the inner product $(\cdot)_\gamma$. Evidently, the multiplicity of each eigenvalue is just $(n-1)$. This is precisely the number of independent polarization states of the electromagnetic field. In terms of the Epstein zeta function (B.1) we see that

$$\zeta(s; \Delta_1^{\mathbb{T}^n}|_{imd_2^*}) = (n-1) E_n(s; \frac{2\pi}{L_1}, \dots, \frac{2\pi}{L_n}). \quad (6.6)$$

The regularized vacuum energy for the transverse modes (4.22) is then given by

$$\varepsilon_{vac}^{reg} := \frac{1}{2} \lim_{\eta \rightarrow 0} \left[\left(\frac{e\mu}{2} \right)^{-2\eta} E_n\left(-\frac{1}{2} - \eta; \frac{2\pi}{L_1}, \dots, \frac{2\pi}{L_n}\right) + \left(\frac{e\mu}{2} \right)^{2\eta} E_n\left(-\frac{1}{2} + \eta; \frac{2\pi}{L_1}, \dots, \frac{2\pi}{L_n}\right) \right]. \quad (6.7)$$

Applying the reflection formula (B.5) and performing the limit $\eta \rightarrow 0$ leads to

$$\varepsilon_{vac}^{reg} = -\frac{n-1}{2} \left[\prod_{j=1}^n L_j \right] \pi^{-\frac{n+1}{2}} \Gamma\left(\frac{n+1}{2}\right) E_n\left(\frac{n+1}{2}; L_1, \dots, L_n\right). \quad (6.8)$$

By construction the vacuum free energy is finite which is - of course - confirmed by the fact that $E_n(\frac{n+1}{2}; L_1, \dots, L_n)$ converges and is positive for every n [53]. Moreover, for any given s , the minima of $E_n(s; L_1, \dots, L_n)$ with fixed volume $V = \prod_{i=1}^n L_i$ appears at $L_1 = \dots = L_n$.

As a consequence, there is no ambiguity in defining a finite vacuum energy. Inserting this result into (4.26), one immediately obtains the expression for the free energy on the n -torus, namely

$$\begin{aligned}
\mathcal{F}^\theta(\beta; V) = & -\frac{n-1}{2} V \pi^{-\frac{n+1}{2}} \Gamma\left(\frac{n+1}{2}\right) E_n\left(\frac{n+1}{2}; L_1, \dots, L_n\right) + \frac{1}{2V} \sum_{j=1}^n L_j^2 \langle \theta_j \rangle^2 \\
& + \frac{n-1}{\beta} \sum_{(k_1, \dots, k_n) \in \mathbb{Z}_0^n} \ln \left[1 - e^{-2\pi\beta \sqrt{\sum_{l=1}^n (\frac{k_l}{L_l})^2}} \right] \\
& - \frac{1}{\beta} \sum_{j=1}^n \ln \Theta_1 \left(\frac{1}{2\pi\sqrt{-1}} \beta \frac{L_j^2}{V} \langle \theta_j \rangle \middle| \frac{\sqrt{-1}}{2\pi} \beta \frac{L_j^2}{V} \right) - \frac{1}{\beta} \sum_{\substack{j,k=1 \\ j < k}}^n \ln \Theta_1 \left(0 \middle| 2\pi\sqrt{-1} \beta \frac{V}{L_j^2 L_k^2} \right).
\end{aligned} \tag{6.9}$$

Notice that $TorH^2(\mathbb{T}^n; \mathbb{Z}) = 0$. For $n = 1$ we recover the free energy of pure electrodynamics on the circle (see e.g. [54]).

For sake of completeness we want to sketch an alternative calculation of the free-energy which derives directly from (3.35). The eigenvalues of $\Delta_p^{\mathbb{T}_\beta^1 \times \mathbb{T}^n} |_{\mathcal{H}^p(\mathbb{T}_\beta^1 \times \mathbb{T}^n)^\perp}$ are given by

$$\begin{aligned}
\nu_{\underline{k}}^{(0,0)} &= \left(\frac{2\pi k_0}{\beta} \right)^2 + \sum_{i=1}^n \left(\frac{2\pi k_i}{L_i} \right)^2, \quad p = 0 \\
\nu_{\underline{k}}^{(1,0)} &= \nu_{\underline{k}}^{(0,1)} = \left(\frac{2\pi k_0}{\beta} \right)^2 + \sum_{i=1}^n \left(\frac{2\pi k_i}{L_i} \right)^2, \quad p = 1
\end{aligned} \tag{6.10}$$

where $\underline{k} = (k_0, k_1, \dots, k_n) \in \mathbb{Z}_0^{n+1} := \mathbb{Z}^{n+1} \setminus 0$. Given the multiplicities of the eigenvalues, the zeta functions of the Laplace operators become proportional to the Epstein zeta functions in $n + 1$ dimensions, namely

$$\zeta(s; \Delta_p^{\mathbb{T}_\beta^1 \times \mathbb{T}^n} |_{\mathcal{H}^p(\mathbb{T}_\beta^1 \times \mathbb{T}^n)^\perp}) = \begin{cases} (n+1) E_{n+1}(s; \frac{2\pi}{\beta}, \frac{2\pi}{L_1}, \dots, \frac{2\pi}{L_n}) & \text{if } p = 1 \\ E_{n+1}(s; \frac{2\pi}{\beta}, \frac{2\pi}{L_1}, \dots, \frac{2\pi}{L_n}) & \text{if } p = 0. \end{cases} \tag{6.11}$$

By applying the duality relation (A.3) to $\Theta_1(\theta_j | \beta^{-1} \cdot)$ in (3.35) and using the Chowla-Selberg formula (B.9) by setting $m = n + 1$, $l = 1$ and $c_1 = \frac{2\pi}{\beta}$, $c_2 = \frac{2\pi}{L_1}, \dots, c_{n+1} = \frac{2\pi}{L_n}$ one immediately obtains (6.9).

In the torus case the vacuum energy is unique and the equation of state is modified only by the Riemann Theta functions. In contrast the free energy of the scalar massless gas transforms properly under scale transformations [23].

In the next two subsections we will derive explicit formulae for the thermodynamic functions in the low- and high temperature regimes, respectively. In order to simplify the calculations we will consider hereafter a *uniform* n -torus, i.e. $L_1 = \dots = L_n = V^{\frac{1}{n}}$.

6.1 The low temperature regime

It is evident that expression (6.9) is suitable for studying the low temperature behavior ($\beta \gg 1$) of the system. Let us indicate this fact by a subscript and write $\mathcal{F}_{low}^\theta(\beta; V)$ when the free energy is displayed in the form (6.9).

We can give an explicit series expansion for the last two terms in (6.9), denoted by f_{low}^θ for further reference. Since $|\langle\theta_j\rangle| \leq \frac{1}{2}$, the constraint (A.5) is satisfied. Eq. (A.6) can be applied to f_{low}^θ giving the following series expansion

$$\begin{aligned} f_{low}^\theta(\beta; V) := & -\frac{1}{\beta} \ln 2^\Lambda - \frac{1}{\beta} \sum_{m=1}^{\infty} \frac{1}{m} \frac{n + 2(-1)^m \Lambda}{1 - e^{\beta V \frac{2-n}{n} m}} \\ & - \frac{1}{\beta} \sum_{\substack{j=1 \\ |\langle\theta_j\rangle| \neq \frac{1}{2}}}^n \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \frac{e^{-\frac{\beta}{2} V \frac{2-n}{n} m(1-2\langle\theta_j\rangle)} + e^{-\frac{\beta}{2} V \frac{2-n}{n} m(1+2\langle\theta_j\rangle)}}{1 - e^{-\beta V \frac{2-n}{n} m}} \\ & - \frac{n(n-1)}{2\beta} \sum_{m=1}^{\infty} \frac{1}{m} \frac{1 + 2(-1)^m e^{\frac{(2\pi)^2}{2} \beta V \frac{n-4}{n} m}}{1 - e^{(2\pi)^2 \beta V \frac{n-4}{n} m}}, \end{aligned} \quad (6.12)$$

where Λ is the number of components θ_j which satisfy $|\langle\theta_j\rangle| = \frac{1}{2}$. Clearly $0 \leq \Lambda \leq n$. Apart from the first term, all others terms decrease exponentially for $\beta \rightarrow \infty$.

The vacuum energy $\mathcal{F}_0^\theta(V)$ is determined by taking the zero temperature limit of (6.9)

$$\begin{aligned} \mathcal{F}_0^\theta(V) := & \lim_{\beta \rightarrow \infty} \mathcal{F}_{low}^\theta(\beta; V) = \\ & = -\frac{n-1}{2} \pi^{-\frac{n+1}{2}} \Gamma\left(\frac{n+1}{2}\right) E_n\left(\frac{n+1}{2}; 1, \dots, 1\right) V^{-\frac{1}{n}} + V^{\frac{2-n}{n}} \sum_{j=1}^n \frac{\langle\theta_j\rangle^2}{2}. \end{aligned} \quad (6.13)$$

If $\vec{\theta} \in \mathbb{Z}^n$, the free energy of the Maxwell field equals the sum of the free energy of $(n-1)$ massless scalar fields (see e.g. the discussion of the massless scalar field with periodic boundary conditions in [17] and [23]). It is evident, that $\mathcal{F}_0^\theta(\lambda^n V) = \lambda^{-1} \mathcal{F}_0^\theta(V)$ for $\vec{\theta} \in \mathbb{Z}^n$.

The internal energy derives from (6.9) and reads for the uniform n -torus

$$\begin{aligned} U_{low}^\theta(\beta; V) = & \frac{(1-n)}{2} V^{-\frac{1}{n}} \pi^{-\frac{n+1}{2}} \Gamma\left(\frac{n+1}{2}\right) E_n\left(\frac{n+1}{2}; 1, \dots, 1\right) + V^{\frac{2-n}{n}} \sum_{j=1}^n \frac{\langle\theta_j\rangle^2}{2} \\ & + \sum_{\vec{k} \in \mathbb{Z}_0^n} \frac{2\pi(n-1)V^{-\frac{1}{n}}|\vec{k}|}{e^{2\pi\beta V^{-\frac{1}{n}}|\vec{k}|} - 1} + \frac{\partial}{\partial\beta} \left(\beta f_{low}^\theta(\beta; V) \right), \end{aligned} \quad (6.14)$$

where we have defined the abbreviations $\vec{k} = (k_1, \dots, k_n)$ and $|\vec{k}| = (\sum_{l=1}^n k_l^2)^{1/2}$. A direct calculation using (6.12) shows that the contributions in the last term of (6.14) decrease exponentially for low temperatures. One obtains

$$U_0^\theta(V) := \lim_{\beta \rightarrow \infty} U_{low}^\theta(\beta; V) = \lim_{\beta \rightarrow \infty} \mathcal{F}_{low}^\theta(\beta; V) = \mathcal{F}_0^\theta(V). \quad (6.15)$$

The entropy of the system is given by

$$\begin{aligned}
S_{low}^\theta(\beta; V) = & (1-n) \sum_{\vec{k} \in \mathbb{Z}_0^n} \ln \left[1 - e^{-2\pi\beta V^{-\frac{1}{n}} |\vec{k}|} \right] + (n-1) V^{-\frac{1}{n}} \beta \sum_{\vec{k} \in \mathbb{Z}_0^n} \frac{2\pi |\vec{k}|}{e^{2\pi\beta V^{-\frac{1}{n}} |\vec{k}|} - 1} \\
& + \beta^2 \frac{\partial}{\partial \beta} f_{low}^\theta(\beta; V).
\end{aligned} \tag{6.16}$$

Using the series expansion (6.12) and performing the zero temperature limit only the first term in (6.12) survives, leading to

$$\lim_{\beta \rightarrow \infty} S_{low}^\theta(\beta; V) = \ln 2^\Lambda. \tag{6.17}$$

This is a consequence of the fact that the ground-state of the system is degenerate of degree 2^Λ . To verify this we notice that for a fixed θ the lowest energy level of \hat{H}_{harm}^θ (see (4.12)) in a given topological (monopole) sector reads

$$\varepsilon_{0, \vec{m}}^\theta = \frac{1}{2} V^{\frac{2-n}{n}} \sum_{j=1}^n \langle \theta_j \rangle^2 + \frac{(2\pi)^2}{2} V^{\frac{n-4}{n}} \sum_{k=1}^{\frac{n(n-1)}{2}} m_k^2. \tag{6.18}$$

Here $\vec{m} = (m_1, \dots, m_k, \dots) \in \mathbb{Z}^{\frac{n(n-1)}{2}}$ labels the topological sectors. The degeneracy appears exactly for the parameter values $\theta_j = \frac{2k_j+1}{2}$, $k_j \in \mathbb{Z}$ (i.e. $|\langle \theta_j \rangle| = \frac{1}{2}$). Hence we have explicitly proved that the 3rd law of thermodynamics (Nernst theorem) holds for the photon gas on the n -torus.

The pressure of the photon gas admits the following form:

$$\begin{aligned}
P_{low}^\theta(\beta; V) = & \frac{V^{-\frac{n+1}{n}}}{n} \left[\sum_{\vec{k} \in \mathbb{Z}_0^n} \frac{2\pi(n-1)|\vec{k}|}{e^{2\pi\beta V^{-\frac{1}{n}} |\vec{k}|} - 1} - \frac{n-1}{2} \pi^{-\frac{n+1}{2}} \Gamma\left(\frac{n+1}{2}\right) E_n\left(\frac{n+1}{2}; 1, \dots, 1\right) \right] \\
& + \frac{n-2}{n} V^{\frac{2(1-n)}{n}} \sum_{j=1}^n \frac{\langle \theta_j \rangle^2}{2} - \frac{\partial}{\partial V} f_{low}^\theta(\beta; V).
\end{aligned} \tag{6.19}$$

In the zero temperature limit, the pressure converges to

$$\begin{aligned}
P_0^\theta(V) &:= \lim_{\beta \rightarrow \infty} P_{low}^\theta(\beta; V) = \\
&= -\frac{n-1}{2n} V^{-\frac{n+1}{n}} \pi^{-\frac{n+1}{2}} \Gamma\left(\frac{n+1}{2}\right) E_n\left(\frac{n+1}{2}; 1, \dots, 1\right) + \frac{n-2}{n} V^{\frac{2(1-n)}{n}} \sum_{j=1}^n \frac{\langle \theta_j \rangle^2}{2}.
\end{aligned} \tag{6.20}$$

For $n \neq 1$, one gets for the large volume limit $\lim_{V \rightarrow \infty} P_0^\theta(V) = 0$. Additionally, the following four cases can be easily distinguished:

a) $n = 1$: $P_0^\theta(V) = -\frac{\langle \theta \rangle^2}{2}$ is non vanishing whenever $\theta \notin \mathbb{Z}$.

- b) $n = 2$:** The pressure is exclusively determined by the (vacuum) transverse modes of the gauge fields. Hence $P_0^\theta(V) < 0$ for all V .
- c) $n = 3$:** For $\vec{\theta} \in \mathbb{Z}^3$ one gets $P_0^\theta(V) < 0$ for all V .
- d) $n > 3$:** For each $\vec{\theta} \notin \mathbb{Z}^n$ there exists a critical volume V_{crit} such that $P_0^\theta(V_{crit}) = 0$. For $V < V_{crit}$ one has $P_0^\theta(V) > 0$ whereas for $V > V_{crit}$ one has $P_0^\theta(V) < 0$.

A direct comparison of (6.15) and (6.20) shows that

$$P_0^\theta(V) = \frac{1}{nV} U_0^\theta(V) + \frac{n-3}{n} V^{\frac{2(1-n)}{n}} \sum_{j=1}^n \frac{\langle \theta_j \rangle^2}{2}. \quad (6.21)$$

Alternatively this result may be obtained as the zero-temperature limit of the equation of state (5.6).

6.2 The high temperature regime

An appropriate expression - hereafter called $\mathcal{F}_{high}^\theta(\beta; V)$ - for analyzing the high-temperature regime ($\beta \ll 1$) can be obtained from (3.35) and (6.11) as follows: Taking $m = n + 1$, $l = n$, $c_1 = \frac{2\pi}{L_1}, \dots, c_n = \frac{2\pi}{L_n}$ and $c_{n+1} = \frac{2\pi}{\beta}$ in (B.9) and applying (A.3) one obtains after a lengthy calculation

$$\begin{aligned} \mathcal{F}_{high}^\theta(\beta; V) = & (1-n)\pi^{-\frac{n+1}{2}} \Gamma\left(\frac{n+1}{2}\right) \zeta_R(n+1) \beta^{-(n+1)} V + \frac{\kappa_1}{\beta} \ln \beta + \frac{1}{\beta} (\kappa_2 \ln V + \kappa_3 \ln 2\pi) \\ & - 2(n-1) \beta^{-\frac{n+2}{2}} V \sum_{k_{n+1}=1}^{\infty} \sum_{\vec{k} \in \mathbb{Z}_0^n} k_{n+1}^{\frac{n}{2}} \left[\sum_{l=1}^n (k_l L_l)^2 \right]^{-\frac{n}{4}} \\ & \times K_{\frac{n}{2}} \left(\frac{2\pi k_{n+1}}{\beta} \sqrt{\sum_{l=1}^n (k_l L_l)^2} \right) + \frac{1-n}{2\beta} E'_n(0; \frac{1}{L_1}, \dots, \frac{1}{L_n}) \\ & - \frac{1}{\beta} \sum_{\substack{j,k=1 \\ j < k}}^n \ln \Theta_1 \left(0 \middle| \frac{\sqrt{-1} \beta^{-1} L_j^2 L_k^2}{2\pi V} \right) - \frac{1}{\beta} \sum_{j=1}^n \ln \Theta_1 \left(\langle \theta_j \rangle \middle| \frac{2\pi \sqrt{-1} \beta^{-1} V}{L_j^2} \right), \end{aligned} \quad (6.22)$$

with the three constants $\kappa_1 = \frac{n^2-3n+4}{4}$, $\kappa_2 = \frac{n^2-7n+8}{4}$ and $\kappa_3 = \frac{n^2-7n+4}{4}$. These constants are non vanishing for all dimensions n .

We want to emphasize that the expressions for both \mathcal{F}_{low}^θ and $\mathcal{F}_{high}^\theta$ are valid for all β and V , that is $\mathcal{F}_{high}^\theta = \mathcal{F}_{low}^\theta$. However, their difference lies in the different polynomial structure as function of the temperature.

Evidently, (A.5) is satisfied so that for the uniform n -torus the sum of the last two terms in (6.22), denoted by $f_{high}^\theta(\beta; V)$, admits the following series expansion

$$f_{high}^\theta(\beta; V) = -\frac{n(n-1)}{2\beta} \left(\sum_{m=1}^{\infty} \frac{1}{m} \frac{1}{1 - e^{m\beta^{-1}V^{\frac{4-n}{n}}}} + 2 \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \frac{e^{\frac{1}{2}m\beta^{-1}V^{\frac{4-n}{n}}}}{1 - e^{m\beta^{-1}V^{\frac{4-n}{n}}}} \right) \\ - \frac{1}{\beta} \left(\sum_{m=1}^{\infty} \frac{1}{m} \frac{1}{1 - e^{(2\pi)^2 m\beta^{-1}V^{\frac{n-2}{n}}}} + 2 \sum_{j=1}^n \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \frac{e^{\frac{(2\pi)^2}{2} m\beta^{-1}V^{\frac{n-2}{n}}} \cos(2\pi m \langle \theta_j \rangle)}{1 - e^{(2\pi)^2 m\beta^{-1}V^{\frac{n-2}{n}}}} \right). \quad (6.23)$$

All terms show an exponential decrease for $\beta \rightarrow 0$. The same is true for the various derivatives of f_{high}^θ which appear in the different thermodynamic functions. Since $K_\nu(z) \simeq \sqrt{\frac{\pi}{2z}} e^{-z} (1 + \mathcal{O}(\frac{1}{z}))$ for $|z| \rightarrow \infty$ [43], the summands in the fourth term of (6.22) show an exponential decrease as $\beta \rightarrow 0$.

As has been stated, (6.22) is exact. But what would be the result, if the general formula (4.34) was used instead of (6.22) for determining the high-temperature limit? Taking into account (6.11) and the following explicit expressions for the Seeley coefficients of the Laplace operator

$$a_m(\Delta_1^{\mathbb{T}^n} |_{imd_2^*}) = \begin{cases} (4\pi)^{-\frac{n}{2}} (n-1) V & \text{if } m = 0 \\ 1 - n & \text{if } m = n \\ 0 & \text{if } m \neq 0, m \neq n, \end{cases} \quad (6.24)$$

we would have obtained (6.22) up to the exponentially decreasing fourth term.

In the high temperature regime the internal energy is given by

$$U_{high}^\theta(\beta; V) = n(n-1)\pi^{-\frac{n+1}{2}} \Gamma\left(\frac{n+1}{2}\right) \zeta_R(n+1)\beta^{-(n+1)} V + \frac{\kappa_1}{\beta} \\ - 4\pi(n-1)\beta^{-\frac{n+4}{2}} V^{\frac{n+2}{2n}} \sum_{k_{n+1}=1}^{\infty} \sum_{\vec{k} \in \mathbb{Z}_0^n} k_{n+1}^{\frac{n+2}{2}} |\vec{k}|^{\frac{2-n}{2}} K_{\frac{n}{2}-1} \left(\frac{2\pi V^{\frac{1}{n}}}{\beta} k_{n+1} |\vec{k}| \right) \quad (6.25) \\ + \frac{\partial}{\partial \beta} \left(\beta f_{high}^\theta(\beta; V) \right),$$

where the relation $\frac{d}{dz} K_\nu(z) = -K_{\nu-1}(z) - \frac{\nu}{z} K_\nu(z)$ have been used (see [43]). Apart from the leading Stefan-Boltzmann term all other terms in (6.25) show a exponential decrease in the high temperature limit. In addition there exists a term linear in the temperature which is positive for all dimensions n . A lengthy calculation finally gives the following expression for the entropy

$$\begin{aligned}
S_{high}^\theta(\beta; V) &= (n^2 - 1)\pi^{-\frac{n+1}{2}} \Gamma\left(\frac{n+1}{2}\right) \zeta_R(n+1)\beta^{-n} V + \kappa_1(1 - \ln \beta) + \\
&+ 2(n-1)\beta^{-\frac{n}{2}} V^{\frac{1}{2}} \sum_{k_{n+1}=1}^{\infty} \sum_{\vec{k} \in \mathbb{Z}_0^n} k_{n+1}^{\frac{n}{2}} |\vec{k}|^{-\frac{n}{2}} K_{\frac{n}{2}} \left(\frac{2\pi V^{\frac{1}{n}}}{\beta} k_{n+1} |\vec{k}| \right) \\
&- 4\pi(n-1)\beta^{-\frac{n+2}{2}} V^{\frac{n+2}{2n}} \sum_{k_{n+1}=1}^{\infty} \sum_{\vec{k} \in \mathbb{Z}_0^n} k_{n+1}^{\frac{n+2}{2}} |\vec{k}|^{\frac{2-n}{2}} K_{\frac{n}{2}-1} \left(\frac{2\pi V^{\frac{1}{n}}}{\beta} k_{n+1} |\vec{k}| \right) \quad (6.26) \\
&+ \frac{n-1}{2} E'_n(0; 1, \dots, 1) - \kappa_3 \ln 2\pi - \kappa_4 \ln V \\
&+ \beta^2 \frac{\partial}{\partial \beta} f_{high}^\theta(\beta; V).
\end{aligned}$$

where $\kappa_4 = \kappa_2 + \frac{n-1}{n}$. The pressure in the high temperature regime finally reads

$$\begin{aligned}
P_{high}^\theta(\beta; V) &= (n-1)\pi^{-\frac{n+1}{2}} \Gamma\left(\frac{n+1}{2}\right) \zeta_R(n+1)\beta^{-(n+1)} - \frac{\kappa_4}{\beta V} \\
&- 4\pi \frac{(n-1)}{n} \beta^{-\frac{n+4}{2}} V^{\frac{2-n}{2n}} \sum_{k_{n+1}=1}^{\infty} \sum_{\vec{k} \in \mathbb{Z}_0^n} k_{n+1}^{\frac{n+2}{2}} |\vec{k}|^{\frac{2-n}{2}} K_{\frac{n}{2}-1} \left(\frac{2\pi V^{\frac{1}{n}}}{\beta} k_{n+1} |\vec{k}| \right) \quad (6.27) \\
&- \frac{\partial}{\partial V} f_{high}^\theta(\beta; V).
\end{aligned}$$

The term linear in temperature and volume is caused by the harmonic part of the gauge field on \mathbb{T}^n . It vanishes for $n = 2$ and is positive for $n = 3, 4$.

7 Summary

To summarize, we want to highlight the main results of this paper: We shed some light onto the relationship between the topology of the compact, closed and connected spatial manifold X and the thermodynamic properties of quantum Maxwell theory at finite temperature. This has been achieved by providing a rigorous determination of the free energy for this system. It has been shown that there are inequivalent quantum theories giving rise to θ -vacua, which are related to the first Betti number of X . The vacuum energy is the sum of the energy of the transverse modes and the energy of the harmonic modes of the Maxwell field. The finite temperature excitations are affected by the topology of X unless both the first and second cohomology of X vanish.

In the Hamilton approach the quantization has been performed directly on the physical phase space using an appropriate parametrization of that space. The minimal subtraction scheme was applied to regularize the vacuum energy of the transverse modes.

The functional integral quantization method on the other hand requires some advanced steps in order to overcome the Gribov problem: We constructed an integrable and globally defined measure on the space of all thermal gauge fields and introduced a consistent reduction procedure to eliminate the gauge degrees of freedom. By expanding the zeta function of the Laplace operators on $\mathbb{T}_\beta^1 \times X$ in terms of the zeta function associated to the corresponding

Laplace operators on the spatial manifold X we succeeded in proving the equivalence with the Hamiltonian approach.

The thermodynamics of a photon gas confined to a n -torus is explicitly elaborated as a special example. Exact expressions for the thermodynamic functions for both the low- and high temperature regimes are derived and the validity of the Nernst theorem is proved.

In light of the ongoing interest in the thermodynamic structure of quantum fields on spaces with non-trivial topologies, it will be a subject of future work to extend the present analysis to manifolds with boundaries and to apply these results for studying further the topological aspects of the Casimir effect in space-time models with compactified extra dimensions.

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A Appendix: Riemann Theta functions

In this section we want to summarize the main facts regarding the Riemann Theta function. Let us now introduce the r -dimensional Riemann Theta function with characteristics $a, b \in \mathbb{R}^r$ by

$$\Theta_r \begin{bmatrix} a \\ b \end{bmatrix} (u|B) = \sum_{m \in \mathbb{Z}^r} \exp \{ \pi \sqrt{-1} (m + a)^\dagger \cdot B \cdot (m + a) + 2\pi \sqrt{-1} (m + a)^\dagger \cdot (u + b) \}, \quad (\text{A.1})$$

for a symmetric complex $r \times r$ dimensional square matrix B whose imaginary part is positive definite, $u \in \mathbb{C}^r$ and the superscript \dagger denotes the transpose. It has the modular property

$$\Theta_r \begin{bmatrix} a + k \\ b + l \end{bmatrix} (u|B) = e^{2\pi \sqrt{-1} a^\dagger l} \Theta_r \begin{bmatrix} a \\ b \end{bmatrix} (u|B), \quad k, l \in \mathbb{Z}^r. \quad (\text{A.2})$$

We write $\Theta_r(u|B) := \Theta_r \begin{bmatrix} 0 \\ 0 \end{bmatrix} (u|B)$ for the Riemann Theta function with characteristics $(0, 0)$.

It can be easily shown that $\Theta_r(-u|B) = \Theta_r(u|B)$ which leads to $\Theta_r \begin{bmatrix} -u \\ 0 \end{bmatrix} (0|B) = \Theta_r \begin{bmatrix} u \\ 0 \end{bmatrix} (0|B)$. By using the Poisson sum formula one can derive the following important "duality" formula

$$\Theta_r(u|B) = \det(-\sqrt{-1}B)^{-\frac{1}{2}} \Theta_r \begin{bmatrix} u \\ 0 \end{bmatrix} (0|-B^{-1}). \quad (\text{A.3})$$

In order to provide a suitable asymptotic expansion of the Riemann Theta function we recall the existence of the infinite product representation (Jacobi triple product) which reads

$$\Theta_1(u|B) = \prod_{k=1}^{\infty} (1 - q^{2k})(1 + q^{2k-1}e^{2\pi\sqrt{-1}u})(1 + q^{2k-1}e^{-2\pi\sqrt{-1}u}), \quad (\text{A.4})$$

where $q = e^{\pi\sqrt{-1}B}$. Let us now take $B = \sqrt{-1}\tau$, with $\tau \in \mathbb{R}$ and $\tau > 0$. If τ and $u = \Re u + \sqrt{-1}\Im u$ satisfy the constraint

$$\tau - 2|\Im u| \geq 0, \quad (\text{A.5})$$

then one can apply the series expansion of the logarithm and the geometric series formula to (A.4). A straightforward calculation finally gives

$$\begin{aligned} \ln \Theta_1(u|\sqrt{-1}\tau) &= - \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \frac{q^{2km}}{m} + \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} q^{(2k-1)m} \left(e^{2\pi\sqrt{-1}mu} + e^{-2\pi\sqrt{-1}mu} \right) \\ &= \sum_{m=1}^{\infty} \frac{1}{m} \frac{1}{1 - e^{2\pi\tau m}} + \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \frac{e^{\pi\tau m} \left(e^{2\pi\sqrt{-1}mu} + e^{-2\pi\sqrt{-1}mu} \right)}{1 - e^{2\pi\tau m}}. \end{aligned} \quad (\text{A.6})$$

B Appendix: Epstein zeta functions

In this appendix the main results on Epstein zeta functions [37, 38] (e.g. see also [23, 36]) are summarized. Let us introduce the homogeneous Epstein zeta function E_m in the variables $(c_1, \dots, c_m) \in \mathbb{R}^m$ by

$$E_m(s; c_1, \dots, c_m) = \sum_{(k_1, \dots, k_m) \in \mathbb{Z}_0^m} \left[\sum_{i=1}^m (c_i k_i)^2 \right]^{-s}, \quad s \in \mathbb{C}, \quad (\text{B.1})$$

where $\mathbb{Z}_0^m := \mathbb{Z}^m \setminus \{0\}$. The sum is convergent for $\Re(s) > \frac{m}{2}$. The analytic continuation of the Epstein zeta function is regular on the whole complex s -plane up to a unique pole at $s = \frac{m}{2}$. Under a constant scale transformation $c_i \mapsto \lambda c_i$, $\lambda \in \mathbb{R}$, the Epstein function transforms as

$$E_m(s; \lambda c_1, \dots, \lambda c_m) = \lambda^{-2s} E_m(s; c_1, \dots, c_m). \quad (\text{B.2})$$

Epstein zeta functions in different dimensions are related by the Chowla-Selberg formula,

$$\begin{aligned} E_m(s; c_1, \dots, c_m) &= E_l(s; c_1, \dots, c_l) + \frac{\pi^{\frac{l}{2}} \Gamma(s - \frac{l}{2})}{\prod_{i=1}^l c_i \Gamma(s)} E_{m-l}(s - \frac{l}{2}; c_{l+1}, \dots, c_m) \\ &\quad + \frac{1}{\Gamma(s)} T_{m,l}(s; c_1, \dots, c_m), \end{aligned} \quad (\text{B.3})$$

where

$$\begin{aligned} T_{m,l}(s; c_1, \dots, c_m) &= \frac{2\pi^s}{\prod_{i=1}^l c_i} \sum_{(k_1, \dots, k_l) \in \mathbb{Z}_0^l} \sum_{(k_{l+1}, \dots, k_m) \in \mathbb{Z}_0^{m-l}} \left[\frac{\sum_{i=1}^l (\frac{k_i}{c_i})^2}{\sum_{i=l+1}^m (k_i c_i)^2} \right]^{\frac{2s-l}{4}} \\ &\quad \times K_{s-\frac{l}{2}} \left(2\pi \sqrt{\left(\sum_{i=1}^l (\frac{k_i}{c_i})^2 \right) \left(\sum_{i=l+1}^m (k_i c_i)^2 \right)} \right). \end{aligned} \quad (\text{B.4})$$

Here $K_\nu(z)$ denotes the modified Bessel function of the second kind [43]. The function $s \mapsto T_{m,l}(s; c_1, \dots, c_m)$ is analytic on \mathbb{C} . Using the Chowla-Selberg formula one can prove the so-called reflection formula

$$\pi^{-s} \Gamma(s) E_m(s; c_1, \dots, c_m) = \frac{\pi^{s-\frac{m}{2}}}{\prod_{i=1}^m c_i} \Gamma(\frac{m}{2} - s) E_m(\frac{m}{2} - s; \frac{1}{c_1}, \dots, \frac{1}{c_m}), \quad (\text{B.5})$$

from which the following results follows

$$E_m(0; c_1, \dots, c_m) = E_1(0; c_1) = 2\zeta_R(0) = -1. \quad (\text{B.6})$$

It follows from (B.2) and (B.6) that the derivative of E_m transforms under constant scale transformations according to

$$E'_m(0; \lambda c_1, \dots, \lambda c_m) = 2 \ln \lambda + E'_m(0; c_1, \dots, c_m), \quad (\text{B.7})$$

where $\zeta_R(s) = \sum_{k=1}^n k^{-s}$ denotes the Riemann zeta function. By recursion we can express the Epstein zeta function in terms of the Riemann zeta function as follows:

$$\begin{aligned} E_m(s; c_1, \dots, c_m) &= 2c_1^{-2s} \zeta_R(2s) + \frac{2}{\Gamma(s)} \sum_{i=1}^{m-1} \frac{\pi^{\frac{i}{2}} \Gamma(s - \frac{i}{2})}{c_{i+1}^{2s-i} \prod_{j=1}^i c_j} \zeta_R(2s - i) \\ &\quad + \frac{4\pi^s}{\Gamma(s)} \sum_{i=1}^{m-1} \frac{1}{\prod_{j=1}^i c_j} \sum_{(k_1, \dots, k_i) \in \mathbb{Z}_0^i} \sum_{k_{i+1}=1}^{\infty} \frac{\left[\sum_{j=1}^i (\frac{k_j}{c_j})^2 \right]^{\frac{s-i}{2}-\frac{i}{4}}}{[k_{i+1} c_{i+1}]^{s-\frac{i}{2}}} \\ &\quad \times K_{s-\frac{i}{2}} \left(2\pi k_{i+1} c_{i+1} \sqrt{\sum_{j=1}^i (\frac{k_j}{c_j})^2} \right). \end{aligned} \quad (\text{B.8})$$

Since $\lim_{s \rightarrow 0} (\frac{d}{ds} \frac{f(s)}{\Gamma(s)}) = 0$ for a well-behaved function $f(s)$ and using the reflection formula one obtains for the derivative of the Epstein zeta function $E'_m(0; c_1, \dots, c_m) := \frac{\partial}{\partial s} E_m(s; c_1, \dots, c_m) |_{s=0}$ in $s = 0$

$$E'_m(0; c_1, \dots, c_m) = E'_l(0; c_1, \dots, c_l) + \frac{\pi^{-\frac{m}{2}} \Gamma(\frac{m}{2})}{\prod_{i=1}^m c_i} E_{m-l}(\frac{m}{2}; \frac{1}{c_{l+1}}, \dots, \frac{1}{c_m}) + T_{m,l}(0; c_1, \dots, c_m). \quad (\text{B.9})$$

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